Long-time Sobolev stability for small solutions of quasi-linear Klein-Gordon equations on the circle

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Abstract

We prove that higher Sobolev norms of solutions of quasi-linear Klein-Gordon equations with small Cauchy data on $\mathbb{S}^1$ remain small over intervals of time longer than the ones given by local existence theory. This result extends previous ones obtained by several authors in the semi-linear case. The main new difficulty one has to cope with is the loss of one derivative coming from the quasi-linear character of the problem. The main tool used to overcome it is a global paradifferential calculus adapted to the Sturm-Liouville operator with periodic boundary conditions.

0 Introduction

We address in this paper the question of long time Sobolev stability for small solutions of nonlinear Klein-Gordon equations on $\mathbb{S}^1$. Let us recall some known results. Consider $V : \mathbb{S}^1 \to \mathbb{R}$ a smooth nonnegative potential and consider u a solution of the equation

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + (V(x) + m^2)u = f(u) \\ &u|_{t=0} = \epsilon u_0 \\ &\partial_t u|_{t=0} = \epsilon u_0, \end{aligned} \tag{0.0.1}$$

where $\epsilon > 0$ is a small parameter, $m \in]0, +\infty[$, f is a nonlinearity vanishing at order $\kappa + 1 \geq 2$ at 0. It is well known that such an equation has a unique $C^0(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, L^2)$ solution if $u_0 \in H^1(\mathbb{S}^1, \mathbb{R}), u_1 \in L^2(\mathbb{S}^1, \mathbb{R})$ and ϵ is small enough. The question is to decide whether, when $u_0 \in H^{s+1}(\mathbb{S}^1, \mathbb{R}), u_1 \in H^s(\mathbb{S}^1, \mathbb{R})$ $(s \gg 1)$, $\|u(t, \cdot)\|_{H^{s+1}} + \|\partial_t u(t, \cdot)\|_{H^s}$ stays bounded over long intervals of time when $\epsilon \to 0$, i.e. over intervals of length $c\epsilon^{-r+1}$ with $r > \kappa + 1$ (the case $r = \kappa + 1$ would correspond to the bound given by local existence theory). The difficulty of

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the problem comes from the fact that on $\mathbb{S}^1$ one does not have any dispersion making decay linear solutions at infinite times, in contrast to what happens for that equation on the real line (We refer to chapter 7 of the book of Hörmander [13] for results and references concerning the nonlinear Klein-Gordon equation on $\mathbb{R}^d$, and to Shatah [15] for the first occurrence in this setting of the normal form method that will play an essential role below).

Bourgain answered the above question for equation (0.0.1) in [5]. He showed that the solutions remain bounded in $H^{s+1} \times H^s$ for intervals of time of length $c\epsilon^{-N}$ for any N, when $s \gg N$, and when the parameter m in (0.0.1) is taken outside a subset of zero measure of $]0, +\infty[$. Bambusi [1] and Bambusi-Grébert [3] obtained later more precise versions of this result (see also the lectures notes of Grébert [12]). Let us mention that, as far as we know, there is no example of solutions which, when m is in the exceptionnal set excluded in the above result, would have an $H^{s+1} \times H^s$ norm blowing up when time goes to infinity. Nevertheless, Bourgain [6] constructed an example of an abstract perturbation of the linear wave equation for which such a blowing up property occurs.

Two natural questions arise: can such results be extended to equations with more general nonlinearities than the one of (0.0.1), and do they hold true in higher dimension? The latter question has been answered affirmatively for equations of type (0.0.1) on the sphere $\mathbb{S}^d$, or more generally on Zoll manifolds, by Bambusi, Grébert, Szeftel and the author in [2]. The former one has been taken up in [9, 10, 11], including in higher dimensions, for equations of type (0.0.1) in which the right hand side is replaced by a general semi-linear non-linearity $f(u, \partial_t u, \partial_x u)$. For such non-linearities, the solution does not in general exist over an interval of time larger that the one given by local existence theory (i.e. $] - c\epsilon^{-\kappa}, c\epsilon^{\kappa}[$ if f vanishes at order $\kappa + 1$ at zero) – see [8] for examples of blowing-up solutions. Nevertheless, a result proved in [9, 10] asserts that if, for instance, f is homogeneous of *even* degree $\kappa + 1$, then the solution of the equation exists and remains bounded in $H^{s+1} \times H^s$ over an interval of time of length $c\epsilon^{-2\kappa}$. The method of proof was similar to the one used by Bourgain [5], Bambusi [1], Bambusi-Grébert [3], the main novelty being its extension to a higher dimensional setting. Our goal in this paper is to address the same question in one space dimension for *quasi-linear* Klein-Gordon equations. As we shall explain below, the semi-linear methods of the above papers break down immediately because of the extra loss of one derivative coming from the quasi-linear nature of the problem. Our main theorem is stated in section 1 below. We shall in this introduction describe our method on the example

$$\begin{aligned} &\Big(D_t - (1 + a(u, \bar{u}))\sqrt{-\Delta + V + m^2}\Big)u = 0 \\ &u|_{t=0} = \epsilon u_0, \end{aligned} \tag{0.0.2}$$

where u_0 is a smooth complex valued function defined on $\mathbb{S}^1$, $\Delta = \frac{d^2}{dx^2}$, and $u \to a(u, \bar{u})$ is a real valued polynomial in $(u, \bar{u})$, homogeneous of odd degree κ. Our aim is to prove existence of the solution, and uniform control of its H^s-norm ($s \gg 1$) by $C\epsilon$, over an interval of time of length $c\epsilon^{-2\kappa}$ (instead of the length $c\epsilon^{-\kappa}$ given by local existence theory). Let us first recall how the corresponding semi-linear result may be proved. Let us take, for simplicity, the case $V \equiv 0$ and consider

$$\begin{aligned} &\Big(D_t - \sqrt{-\Delta + m^2}\Big)u = f(u, \bar{u}) \\ &u|_{t=0} = \epsilon u_0, \end{aligned} \tag{0.0.3}$$

where $f(u,\bar u)=u^p\bar u^q$ with $p+q=\kappa+1$. Set $\Lambda_m=\sqrt{-\Delta+m^2}$, $\Lambda=\sqrt{-\Delta+1}$ and let Π_n be the spectral projector on the space generated by the eigenfunctions $e^{\pm inx}$ ($n\in\mathbb{N}$). Then the H^s norm is given by $\|u\|^2_{H^s}=\langle\Lambda^s u,\Lambda^s u\rangle=\sum_{n=0}^{+\infty}(1+n^2)^s\|\Pi_n u\|^2_{L^2}$ and if u solves (0.0.3)

$$\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|^2_{H^s}=-\mathrm{Im}\,[\langle\Lambda^s(\Lambda_m u),\Lambda^s u\rangle+\langle\Lambda^s f(u,\bar u),\Lambda^s u\rangle]. \tag{0.0.4}$$

The first term in the right hand side vanishes by self-adjointness of Λ_m, and the second one may be written $-\mathrm{Im}\,M_0(u,\dots,\bar u)$ with
(0.0.5)

$$M_0(\underbrace{u,\dots,u}_{p},\underbrace{\bar u,\dots,\bar u}_{q+1})=\sum_{n_1,\dots,n_{p+q+1}}(1+n^2_{p+q+1})^s\int_{\mathbb{S}^1}\Pi_{n_1}u\cdots\Pi_{n_p}u\overline{\Pi_{n_{p+1}}u}\cdots\overline{\Pi_{n_{p+q+1}}u}\,dx.$$

The idea of the method is to perturb the H^s energy of u by a multilinear expression

$$\mathrm{Re}\,M_1(\underbrace{u,\dots,u,\bar u,\dots,\bar u}_{p+q+1=\kappa+2})$$

such that $\frac{d}{dt}M_1(u,\dots,\bar u)$ will cancel out (0.0.5) up to a remainder which will be $O(\|u\|^{2\kappa+2}_{H^s})$. This gain on the order of vanishing at 0, versus the one of the last term in (0.0.4), allows one to obtain the longer interval of time $c\epsilon^{-2\kappa}$ by standart arguments. Using (0.0.3), one finds that

$$\frac{d}{dt}M_1(u,\dots,u,\bar u,\dots,\bar u)=iL(M_1)(u,\dots,u,\bar u,\dots,\bar u)+R(u,\bar u) \tag{0.0.6}$$

where
(0.0.7)

$$L(M_1)(u,\dots,\bar u)=\sum_1^p M_1(u,\dots,\Lambda_m u,\dots,u,\bar u,\dots,\bar u)-\sum_{p+1}^{p+q+1}M_1(u,\dots,u,\bar u,\dots,\Lambda_m\bar u,\dots,\bar u),$$

and $R(u,\bar u)$ is a remainder obtained substituting $if(u,\bar u)$ to one of the arguments of M_1. Since f contains no derivative of u, $R(u,\bar u)=O(\|u\|^{2\kappa+2}_{H^s})$ as wanted. As $\Lambda_m\Pi_n u=\sqrt{m^2+n^2}\Pi_n u$, one may write
(0.0.8)

$$L(M_1)(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+q+1}}u_{p+q+1})=F_m(n_1,\dots,n_{p+q+1})M_1(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+q+1}}u_{p+q+1}),$$

where we denoted

$$F_m(n_1,\dots,n_{p+q+1})=\sum_1^p\sqrt{m^2+n_j^2}-\sum_{p+1}^{p+q+1}\sqrt{m^2+n_j^2}. \tag{0.0.9}$$

To eliminate in $\frac{d}{dt}[\frac12\|u(t,\cdot)\|^2_{H^s}+\mathrm{Re}\,M_1(u,\dots,\bar u)]$ terms homogeneous of degree $\kappa+1$, one has to choose M_1 so that $L(M_1)=-M_0$ i.e. according to (0.0.8) and (0.0.5)

$$\begin{aligned}M_1(\Pi_{n_1}u_1,\dots,&\Pi_{n_{p+q+1}}u_{p+q+1})=\\&-F_m(n_1,\dots,n_{p+q+1})^{-1}(1+n^2_{p+q+1})^s\int_{\mathbb{S}^1}\Pi_{n_1}u_1\cdots\Pi_{n_{p+q+1}}u_{p+q+1}\,dx.\end{aligned} \tag{0.0.10}$$

Since $p+q$ is even, it may be proved that for m outside an exceptionnal subset of zero measure, $F_m(n_1,\dots,n_{p+q+1})$ does not vanish, and actually

$$|F_m(n_1,\dots,n_{p+q+1})|^{-1} \leq C\mu(n_1,\dots,n_{p+q+1})^{N_0}$$

for some N_0, $\mu(n_1,\dots,n_{p+q+1})$ standing for the third largest among $n_1,\dots,n_{p+q+1}$. This shows that $|F_m|^{-1}$ is bounded from above by a power of a small frequency, which allows one to prove, combining this with convenient estimates of the integral in (0.0.10), that M_1 is a continuous multilinear form on $H^s \times \cdots \times H^s$ for $s \gg N_0$, and so a small perturbation of the H^s energy when u is small. Let us notice that related ideas are used for problems on $\mathbb{R}^n$ by Colliander, Keel, Staffilani, Takaoka and Tao in [7].

Let us go back to the quasi-linear equation (0.0.2). In this case (0.0.4) will write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|^2_{H^s} &= -\mathrm{Im}\,\langle \Lambda^s a(u,\bar u)\Lambda_m u, \Lambda^s u\rangle \\ &= \frac{1}{2i}\langle \Lambda^s[\Lambda^{-2s}\Lambda_m, a\Lambda^{2s}]u, \Lambda^s u\rangle.\end{aligned} \tag{0.0.11}$$

Since the operator $[\Lambda^{-2s}\Lambda_m, a\Lambda^{2s}]$ is of order 0, we still get a quantity well defined on H^s, even if its expression is now a little bit more complicated than (0.0.5). We would like to argue as above and find a new contribution $\mathrm{Re}\, M_1$ to add to $\frac{1}{2}\|u(t,\cdot)\|^2_{H^s}$, so that its time derivative would cancel out the right hand side of (0.0.11), up to remainders. The $R(u,\bar u)$ terms in (0.0.6) would be given by

$$\begin{aligned}R(u,\bar u) = i\Big(&\sum_1^p M_1(u,\dots,u,a(u,\bar u)\Lambda_m u, u,\dots,u,\bar u,\dots,\bar u) \\ &- \sum_{p+1}^{p+q+1} M_1(u,\dots,u,\bar u,\dots,\bar u, a(u,\bar u)\Lambda_m \bar u,\bar u,\dots,\bar u)\Big).\end{aligned} \tag{0.0.12}$$

This quantity is no longer of order 0 in $u,\bar u$ for a general M_1, which means that $R(u,\bar u)$ could no longer be estimated by $C\|u\|^{2\kappa+2}_{H^s}$ but only by $C\|u\|^{2\kappa+1}_{H^s}\|u\|_{H^{s+1}}$. This loss of derivative, which is systematic in quasi-linear problems, cannot be recovered if M_1 is a multilinear form which does not satisfy any structure condition. On the other hand, if we know that M_1 has a structure similar to the quantity in the right hand side of (0.0.11), we may hope to make appear a commutator that will kill the extra loss of one derivative. This is actually the usual way of getting quasi-linear energy inequalities. The price we have to pay to be able to do so is that we must get for M_0, M_1 expressions more explicit that just multilinear quantities satisfying convenient estimates, like those used in the semilinear problems treated in the aforementionned references. We must be able to write M_0 or M_1 as

$$\langle \mathrm{Op}(c(u,\dots,\bar u;\cdot))u, u\rangle$$

where $c(u_1,\dots,u_p;\cdot)$ will be a convenient paradifferential symbol, that may be computed from the equation, and $\mathrm{Op}(c)$ is the operator associated to that symbol. The difficulty that arises is the following: we must work globally on $\mathbb{S}^1$, and cannot restrict ourselves to open subsets of $\mathbb{R}$ through local charts. This is because our class of symbols will have to contain functions defined in terms of $F_m(n_1, \dots, n_{p+q+1})^{-1}$, where F_m is given in (0.0.9) (to be able to construct the analogous of M_1 – see (0.0.10)). This quantity is well defined for m outside an exceptional subset, only when the arguments $n_1, \dots, n_{p+q+1}$ stay in a *discrete set*. In other words, we cannot use Bony's calculus of paradifferential operators on $\mathbb{R}$ [4], since their symbols are functions of a *continuous* phase variable. We must instead define a global paradifferential calculus on $\mathbb{S}^1$, in terms of symbols whose phase variable varies in the (discrete) spectrum of $-\frac{d^2}{dx^2} + V$ on $\mathbb{S}^1$. When $V \equiv 0$, this is done through Fourier series expansions. An example of the type of symbols we have to consider is given by

$$(n_0, n_1) \to \langle a e^{in_0 x}, e^{in_1 x} \rangle = \hat{a}(n_1 - n_0)$$

where $a \in C^\infty(\mathbb{S}^1)$. Such a quantity is rapidly decaying in $n_0 - n_1$, and its $\partial_{n_0} + \partial_{n_1}$ derivative vanishes. In general, when $V \not\equiv 0$, the class of symbols we want to consider has to include quantities like

$$(n_0, n_1) \to \langle a\varphi_{n_0}, \varphi_{n_1} \rangle,$$

where $\varphi_{n_0}, \varphi_{n_1}$ are two eigenfunctions, and we want them to verify estimates of form

$$|(\partial_{n_0} + \partial_{n_1})^\gamma \langle a\varphi_{n_0}, \varphi_{n_1} \rangle| \le C_N \langle n_0 - n_1 \rangle^{-N} (n_0 + n_1)^{-\gamma}. \tag{0.0.13}$$

The first section of this paper is devoted to the construction of *nice basis* of $L^2(\mathbb{S}^1)$, i.e. of orthonormal basis of almost eigenfunctions for which estimates of form (0.0.13) hold true. This is done using quasi-modes for $-\frac{d^2}{dx^2} + V$ which resemble the imaginary exponentials of the free case.

The second section of the paper is devoted to the definition of paradifferential operators associated to symbols whose phase argument varies in a discrete set. We establish the main symbolic calculus properties of such operators.

The third section presents a special class of pseudo-differential operators, containing the operators involved in the writing of equation (0.0.1). These special operators enjoy more explicit symbolic calculus properties that the general ones defined in section 2.

The fourth section is devoted to the proof of the theorem, using the machinery of sections 2 and 3 to be able to get the energy estimates we alluded to at the beginning of this introduction. We first perform a paradifferential diagonalization of the principal part of the wave operator, reducing (0.0.1) to a paradifferential version of (0.0.2). We then apply the energy method, as explained after (0.0.11). The fact that we reduced ourselves to a diagonal principal symbol, together with the symbolic calculus constructed in the preceding sections, allows us to show that the remainders of form (0.0.12) that we get actually involve commutators compensating the apparent loss of one derivative displayed by (0.0.11). In that way, we are able to obtain energy inequalities of type $\frac{d}{dt}\|u(t,\cdot)\|_{H^s}^2 \le C\|u(t,\cdot)\|_{H^s}^{2\kappa+2}$, which imply the long time existence result we are looking for.

Let us conclude this introduction expressing our gratitude to Dario Bambusi for several conversations about this work. Let us say also that we shall use in the text the following notation: we write $n_0 \sim n_1$ to mean that there is a (large) constant $C > 0$ with $C^{-1} n_0 \le n_1 \le C n_0$ when $n_0, n_1 \to +\infty$, and we set $n_0 \ll n_1$ to say that there is a small $c > 0$ with $n_0 \le c n_1$ when $n_0, n_1 \to +\infty$.

1 Main results and nice basis

1.1 Statement of main theorem

We shall be interested in this paper in solutions of the periodic one dimensional quasi-linear Klein-Gordon equation. We denote by $\Delta = \frac{d^2}{dx^2}$ the Laplace operator on $\mathbb{S}^1$, and take $V : \mathbb{S}^1 \to \mathbb{R}_+$ a smooth nonnegative potential. We shall sometimes identify $\mathbb{S}^1$ with the interval $[-\pi, \pi]$ with periodic boundary conditions. We consider a polynomial map

$$\begin{aligned} c : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (X_0, X_1, X_2) &\longrightarrow c(X_0, X_1, X_2) \end{aligned} \tag{1.1.1}$$

which may be written

$$c(X_0, X_1, X_2) = \sum_{k=\kappa}^{\kappa_1} c_k(X_0, X_1, X_2) \tag{1.1.2}$$

where c_k is homogeneous of degree k in (X_0, X_1, X_2). We denote by r the largest odd integer satisfying $\kappa \leq r - 1 \leq 2\kappa$ and

(1.1.3) for any even integer $2k$ satisfying $\kappa \leq 2k < r - 1$, one has $c_{2k}(X_0, X_1, X_2) \equiv 0$.

We shall consider the following equation, where $m > 0$ is a parameter

$$\begin{aligned} &\partial_t^2 v + (1 + c(v, \partial_t v, \partial_x v))^2 [-\Delta + V + m^2] v = 0 \\ &v|_{t=0} = \epsilon v_0 \\ &\partial_t v|_{t=0} = \epsilon v_1, \end{aligned} \tag{1.1.4}$$

where v_0 and v_1 are smooth real valued functions defined on $\mathbb{S}^1$, and $\epsilon > 0$ is a small parameter. Our main result is the following:

Theorem 1.1.1 *There is a zero measure subset $\mathcal{N}$ of $]0, +\infty[$, and for every $m \in \mathcal{N}$, there are $c > 0, s_0 \in \mathbb{N}$, such that for any $s \geq s_0$, any $(v_0, v_1) \in H^{s+1}(\mathbb{S}^1, \mathbb{R}) \times H^s(\mathbb{S}^1, \mathbb{R})$, verifying for $\epsilon \in]0, 1[$*

$$\|v_0\|_{H^{s_0+1}} + \|v_1\|_{H^{s_0}} < \epsilon, \tag{1.1.5}$$

equation (1.1.4) has a unique solution

$$v \in C^0(] - T_\epsilon, T_\epsilon[, H^{s+1}(\mathbb{S}^1, \mathbb{R})) \cap C^1(] - T_\epsilon, T_\epsilon[, H^s(\mathbb{S}^1, \mathbb{R}))$$

with $T_\epsilon \geq c\epsilon^{-r+1}$. Moreover, there is for any $s \geq s_0$ a constant $c_s > 0$, such that if (v_0, v_1) satisfies (1.1.5) with s_0 replaced by s, $\|v(t, \cdot)\|_{H^{s+1}} + \|\partial_t v(t, \cdot)\|_{H^s}$ is uniformly bounded on the interval $] - T'_\epsilon, T'_\epsilon[$ with $T'_\epsilon \geq c_s \epsilon^{-r+1}$.

Remarks • It is enough to prove that for s_0 large enough, condition (1.1.5) with $\epsilon > 0$ small enough implies the existence of an $H^{s_0+1} \times H^{s_0}$ bounded solution defined on $]-T_\epsilon, T_\epsilon[\times \mathbb{S}^1$. We know then that if the Cauchy data (v_0, v_1) belong to $H^{s+1} \times H^s$ with $s \geq s_0$, their smoothness will be propagated by the equation.

• The time of existence given by local existence theory is $c\epsilon^{-\kappa}$. If κ is even and $c_\kappa \not\equiv 0$ in (1.1.2), then (1.1.3) gives $r = \kappa + 1$, and the theorem is empty: it just asserts that there is a solution defined on the interval of time given by local existence theory. Because of that, we shall assume in the sequel that κ is odd.

• If κ is odd, and $c_{2k} \equiv 0$ if $\kappa < 2k < 2\kappa$, we may take $r = 2\kappa + 1$, and we get a solution on an interval of length $\epsilon^{-2\kappa}$, i.e. on a much larger interval than the one given by local existence theory.

• In the semi-linear case, theorem 1.1.1 has been proved (with more general assumptions on the nonlinearity) in [9, 10] when the equation is posed more generally on $\mathbb{S}^d$, or on a Zoll manifold of any dimension.

• For semi-linear equations on Zoll manifolds, whose nonlinearities depend only on v, and not on its derivatives, it has been proved in [2] that the solution of the problem is almost global, i.e. defined on intervals of length $c_N\epsilon^{-N}$ for any N. Moreover one has uniform Sobolev estimates on such intervals. This result had been obtained previously in one dimension by Bourgain [5], on a slightly weaker form, and by Bambusi [1] and Bambusi-Grébert [3].

• In the quasi-linear case, no result seems to have been known, except in the much simpler case of equations of form (1.1.4) with zero potential and a quadratic nonlinearity on $\mathbb{T}^d (d \geq 1)$: see [9]. For such operators and nonlinearities, most of the difficulties we shall encounter in this paper disappear. Actually, the fact that the potential is zero allows one to use Fourier series, and so harmonic analysis. The combination of this and of the fact that the nonlinearity is quadratic makes functions of type (0.0.9) always nonzero *whatever the value of parameter m* on the relevant set of arguments. Because of that, the proof does not use the structure of the spectrum of the Laplacian, and this explains why one is able to treat also the case of tori of higher dimension. On the other hand, as soon as either the potential is nonzero, or the nonlinearity vanishes at order strictly larger than two, the structure of the spectrum plays an essential role. This explains why, in such cases, no result is known on $\mathbb{T}^d (d \geq 2)$, even for semi-linear equations.

• A natural question is to know if theorem 1.1.1 may be extended from $\mathbb{S}^1$ to $\mathbb{S}^d$, as its semi-linear counterpart. We are unable to perform such an extension. This is related to the existence of "nice basis" which will be addressed in next subsection.

1.2 Nice basis

Let $V : \mathbb{S}^1 \to \mathbb{R}_+$ be a smooth function. The large eigenvalues of $-\frac{d^2}{dx^2} + V$ are arranged in couples $(\omega_n^-)^2 \leq (\omega_n^+)^2$, where ω_n^+ and ω_n^- have when $n \to +\infty$ a same asymptotic expansion at any order of form

$$n + \frac{1}{4\pi n}\int_{\mathbb{S}^1} V(x)\,dx + \frac{\alpha_3}{n^3} + \frac{\alpha_5}{n^5} + \cdots \tag{1.2.1}$$

(see for instance the book of Marchenko [14]). We shall denote in this subsection for n large enough by E_n the subspace of $L^2(\mathbb{S}^1, \mathbb{R})$ spanned by the eigenfunctions associated to $(\omega_n^-)^2$ and $(\omega_n^+)^2$, and by Π_n the spectral projection of L^2 onto that subspace. We shall choose a function $\lambda \to \omega(\lambda)$, which is a symbol of order 1, having when $\lambda \to +\infty$ the expansion (1.2.1) (with n replaced by λ). If we write $a_n = O(n^{-\infty})$ to mean that for any $N \in \mathbb{N}$ there is $C_N > 0$ with $|a_n| \leq C_N n^{-N}$, then $\omega(n) - \omega_n^{\pm} = O(n^{-\infty})$. Consequently, we have

$$\|\sqrt{-\Delta + V}\Pi_n - \omega(n)\Pi_n\|_{\mathcal{L}(L^2, L^2)} = O(n^{-\infty}). \tag{1.2.2}$$

Our goal is to construct a basis of each E_n such that some scalar products involving elements of these basis will have symbolic behaviour relatively to the spectral parameters. Before stating the theorem, let us introduce the following notations. For $\tau \in \mathbb{N}$, we denote by $\mathbb{N}_\tau = \{n \in \mathbb{N}; n \geq \tau\}$. If $a : \mathbb{N}_\tau \to \mathbb{C}$ is given, we extend it by 0 to a function defined on $\mathbb{Z}$, and we define $\partial a : \mathbb{N}_\tau \to \mathbb{C}$ by

$$\partial a(n) = a(n+1) - a(n). \tag{1.2.3}$$

We denote by ∂^* the formal adjoint of ∂ for the scalar product $\langle a, b\rangle = \sum_{n\geq\tau} a(n)\overline{b(n)}$, that is

$$\partial^* a(n) = -\partial a(n-1). \tag{1.2.4}$$

We have then for a function a defined on $\mathbb{N}_\tau \times \mathbb{N}_\tau$

$$(\partial_n - \partial^*_{n'})a(n,n') = a(n+1,n') - a(n,n'-1). \tag{1.2.5}$$

We shall use below the following elementary formulas. For a function $a(n)$, denote if $k \in \mathbb{Z}$ $\tau_k a(n) = a(n-k)$. One has then

$$\begin{aligned}
\partial_n(ab) &= (\partial_n a)(\tau_{-1}b) + a(\partial_n b)\\
\partial_n^*(ab) &= (\partial_n^* a)b + (\tau_1 a)(\partial_n^* b)\\
\partial_n(ab) &= (\partial_n a)b + a(\partial_n b) + (\partial_n a)(\partial_n b)\\
\partial_n^*(ab) &= (\partial_n^* a)b + a(\partial_n^* b) + (\partial_n^* a)(\partial_n^* b).
\end{aligned} \tag{1.2.6}$$

Moreover, if we consider functions $a(n,n'), b(n,n')$ defined on $\mathbb{N}_\tau \times \mathbb{N}_\tau$, and if τ_k^1, τ_k^2 are the translation operators relatively to the first and second variable respectively, we have

$$\begin{aligned}
(\partial_n - \partial^*_{n'})(ab) &= (\tau^1_{-1}a)((\partial_n - \partial^*_{n'})b) + ((\partial_n - \partial^*_{n'})a)(\tau^2_1 b)\\
(\partial_n - \partial^*_{n'})(ab) &= a((\partial_n - \partial^*_{n'})b) + ((\partial_n - \partial^*_{n'})a)b + (\partial_n a)(\partial_n b) - (\partial^*_{n'}a)(\partial^*_{n'}b),
\end{aligned} \tag{1.2.7}$$

$$\begin{aligned}
\partial_n[a(n,n)] &= ((\partial_n - \partial^*_{n'})a)(n, n+1)\\
\partial_n^*[a(n,n)] &= -((\partial_n - \partial^*_{n'})a)(n-1, n).
\end{aligned} \tag{1.2.8}$$

Remind that a pseudo-differential operator T, of order 0 on $\mathbb{S}^1$, may be written when acting on a periodic function u as

$$Tu(x) = \int_{\mathbb{S}^1} \sum_{n\in\mathbb{Z}} e^{in(x-y)} a(x,n) u(y)\,dy \tag{1.2.9}$$

where a is a smooth function on $\mathbb{S}^1 \times \mathbb{Z}$, satisfying for any $\alpha, \beta \in \mathbb{N}$,

$$|\partial_x^\alpha \partial_n^\beta a(x,n)| \le C_{\alpha,\beta}(1+|n|)^{-\beta} \tag{1.2.10}$$

(where ∂_x means a usual derivative, and ∂_n is defined by (1.2.3)). We set

$$|a|_P = \sup_{0\le\alpha\le P} \sup_{0\le\beta\le P} \sup_{(x,n)\in\mathbb{S}^1\times\mathbb{Z}} (1+|n|)^\beta |\partial_x^\alpha \partial_n^\beta a(x,n)|. \tag{1.2.11}$$

We may also use a local representation: Let $\chi \in C_0^\infty(\mathbb{R})$ be supported inside an interval of length strictly smaller that 2π. Take $\tilde{\chi} \in C_0^\infty(\mathbb{C})$, $\tilde{\chi} \equiv 1$ close to 0, $\operatorname{Supp}\tilde{\chi}$ small enough and set $\tilde{\chi}_0 = 1 - \tilde{\chi}$. Define

$$\begin{aligned} \tilde{a}(x,\xi) &= \sum_{n=-\infty}^{+\infty} a(x,n)\Theta(x,\xi-n) \\ K(x,y) &= \sum_{n=-\infty}^{+\infty} e^{in(x-y)}\tilde{\chi}_0(e^{i(x-y)}-1)a(x,n) \end{aligned} \tag{1.2.12}$$

with

$$\Theta(x,\eta) = \int e^{-i(x-y)\eta}\tilde{\chi}(e^{i(x-y)}-1)\chi(y)\,dy.$$

Then we have if $\operatorname{Supp} u$ is contained in the domain where $\chi \equiv 1$

$$\begin{aligned} Tu(x) &= \frac{1}{2\pi}\int e^{ix\xi}\tilde{a}(x,\xi)\hat{u}(\xi)\,d\xi + Ru(x) \\ Ru(x) &= \int K(x,y)u(y)\,dy. \end{aligned} \tag{1.2.13}$$

If we set $\tilde{\chi}_{k+1}(z) = z^{-1}\tilde{\chi}_k(z)$, we see that

$$\begin{aligned} K(x,y) &= \sum_n \big(e^{i(n+1)(x-y)} - e^{in(x-y)}\big)\tilde{\chi}_1(e^{i(x-y)}-1)a(x,n) \\ &= \sum_n e^{in(x-y)}\tilde{\chi}_1(e^{i(x-y)}-1)\partial_n^* a(x,n) \\ &= \sum_n e^{in(x-y)}\tilde{\chi}_k(e^{i(x-y)}-1)(\partial_n^*)^k a(x,n). \end{aligned}$$

This shows that K is a smooth 2π-periodic function of (x,y), whose derivatives up to order N are bounded in L^∞ in terms of the constants $C_{\alpha\beta}$ of (1.2.10) for $\alpha+\beta \le N+2$. Moreover, if $x \in [-\pi,\pi]$ and $\operatorname{Supp}\tilde{\chi}$ has been taken small enough, we see that

$$\partial_\eta\Theta(x,\eta) = \int e^{-i(x-y)\eta}(e^{i(x-y)}-1)\tilde{\chi}^1(x,y)\chi(y)\,dy$$

where $\tilde{\chi}^1(x,y) = -i(x-y)(e^{i(x-y)}-1)^{-1}\tilde{\chi}(e^{i(x-y)}-1) \in C^\infty$ if $y \in \operatorname{Supp}\chi \Subset]-\pi,\pi[$, $x \in [-\pi,\pi]$. Consequently $\partial_\eta\Theta(x,\eta) = \Theta^1(x,\eta-1) - \Theta^1(x,\eta)$, for a function Θ^1, of the same form as Θ, satisfying $|\partial_x^\alpha\Theta^1(x,\eta)| \le C_N\langle\eta\rangle^{-N}$ for any α, any N. We may thus write

$$\partial_\xi\tilde{a}(x,\xi) = \sum_n a(x,n)\partial_n[\Theta^1(x,\xi-n)] = \sum_n (\partial_n^* a)(x,n)\Theta^1(x,\xi-n).$$

Computing in the same way higher order derivatives, we get that $\tilde{a}$ is a symbol on $[-\pi,\pi]\times\mathbb{R}$, whose semi-norms are controlled in terms of the corresponding semi-norms of a.

Our aim is to prove the following:

Theorem 1.2.1 *There is $\tau\in\mathbb{N}^*$ and for any $n\geq\tau$, there is an orthonormal basis $(\varphi_n^1,\varphi_n^2)$ of E_n, satisfying the following property: there is $\nu\in\mathbb{R}_+$ and for any $N,\alpha,\beta,\gamma\in\mathbb{N}$ there is a constant $C>0$, such that for any pseudo-differential operator of order 0 on $\mathbb{S}^1$, T, of symbol a, for any $n,n'\in\mathbb{N}_\tau$, any $j,j'\in\{1,2\}$, one has*

$$\left|\partial_n^\alpha(\partial_{n'}^*)^\beta(\partial_n-\partial_{n'}^*)^\gamma\langle\varphi_n^j,T\varphi_{n'}^{j'}\rangle\right|\leq C\langle n-n'\rangle^{-N}(n+n')^{-\gamma}|a|_{\nu+N+\alpha+\beta+\gamma}. \tag{1.2.14}$$

An hilbertian basis $(\varphi_n^j)_{j,n}$ of $L^2(\mathbb{S}^1,\mathbb{R})$, such that (1.2.14) is satisfied for $n,n'\geq\tau$ large enough, will be called a nice basis*.*

Remark The functions φ_n^1,φ_n^2 of the statement are not assumed to be eigenfunctions of $-\Delta+V$. Nevertheless, because of (1.2.2), they verify $\|(\sqrt{-\Delta+V}-\omega(n))(\varphi_n^j)\|_{L^2}=O(n^{-\infty})$.

Before starting the proof of the theorem, let us state a corollary.

Corollary 1.2.2 *Let $(\varphi_n^j)_{j,n}$ be a nice basis of $L^2(\mathbb{S}^1,\mathbb{R})$. Let T_1,T_2 be two pseudo-differential operators of order 0 on $\mathbb{S}^1$. There is $\nu\in\mathbb{R}_+$, and for any $N,\alpha,\beta,\gamma\in\mathbb{N}$, there is $C>0$ such that for any C^∞ function a on $\mathbb{S}^1$, one has*

$$|\partial_n^\alpha(\partial_{n'}^*)^\beta(\partial_n-\partial_{n'}^*)^\gamma\langle T_1\varphi_n^j,a(x)T_2\varphi_{n'}^{j'}\rangle|\leq C\langle n-n'\rangle^{-N}(n+n')^{-\gamma}\sum_{k=0}^{\alpha+\beta+\gamma+N+\nu}\|\partial^k a\|_{L^\infty} \tag{1.2.15}$$

for any $n,n'\in\mathbb{N}^*$.

The corollary follows from (1.2.14) applied to $T=T_1^*aT_2$, which is a pseudo-differential operator of order 0, whose symbol semi-norms $|\cdot|_P$ are controlled in terms of $\|\partial^k a\|_{L^\infty}$ for $k\leq P+\nu_0$, for a fixed $\nu_0\in\mathbb{N}$.

We shall first construct quasi-modes satisfying convenient properties.

Proposition 1.2.3 *There exists for $n\geq\tau$ large enough, functions $\underline{U}_n\in C^\infty([-\pi,\pi],\mathbb{C})$ satisfying the following properties:*

(i) For any $n\in\mathbb{N}_\tau$, any $k\in\mathbb{N}$, $\|\underline{U}_n\|_{L^2[-\pi,\pi]}=1$ and $\partial_x^k\underline{U}_n(\pi)-\partial_x^k\underline{U}_n(-\pi)=O(n^{-\infty}),n\to+\infty$.

(ii) Let T be a pseudo-differential operator of order 0 on $\mathbb{S}^1$. Denote by $U_n(x)$ the function on $\mathbb{R}$ obtained by 2π-periodization of $\underline{U}_n$. Consider U_n as an element of $L^2(\mathbb{S}^1,\mathbb{C})$, and define for $n,n'\in\mathbb{N}_\tau$

$$I_-(n,n')=\langle TU_n,U_{n'}\rangle,\ I_+(n,n')=\langle TU_n,\overline{U}_{n'}\rangle. \tag{1.2.16}$$

There is $\nu \in \mathbb{R}_+$, and for any $\alpha, \beta, \gamma, N \in \mathbb{N}$, a constant $C > 0$ such that, for any operator T as above, defined in terms of a symbol a by (1.2.9), one has

$$|\partial_n^\alpha (\partial_{n'}^*)^\beta (\partial_n - \partial_{n'}^*)^\gamma I_-(n,n')| \leq C \langle n - n' \rangle^{-N} (n+n')^{-\gamma} |a|_{\nu+N+\alpha+\beta+\gamma}, \tag{1.2.17}$$

$$|\partial_n^\alpha (\partial_{n'}^*)^\beta (\partial_n - \partial_{n'}^*)^\gamma I_+(n,n')| \leq C (n+n')^{-N-\gamma} |a|_{\nu+N+\alpha+\beta+\gamma} \tag{1.2.18}$$

for any $n, n' \in \mathbb{N}_\tau$ with $|n - n'| \leq \frac{1}{2}(n+n')$.

(iii) There is a sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathbb{R}_+^*$ such that $h_n^{-1} - \omega(n) = O(n^{-3})$ and

$$\|(-\Delta + V - h_n^{-2}\mathrm{Id})U_n\|_{H^{-2}} = O(n^{-\infty}),\ \|U_n\|_{H^{1/2-\delta}} \leq C_\delta h_n^{-1} \tag{1.2.19}$$

for any $n \geq \tau, \delta > 0$.

We shall first construct $\underline{U}_n$ such that (i) and (iii) hold true.

Lemma 1.2.4 *There are $\delta_0 > 0$ and smooth functions $(x,h) \to \theta(x,h)$, $(x,h) \to b(x,h)$ defined on $[-\pi,\pi]\times[0,\delta_0]$, real valued, even in h, and a sequence $(h_n)_n$ of points of $]0,1]$, with asymptotic expansion*

$$h_n = \frac{1}{n} - \frac{1}{4\pi n^3}\int_{-\pi}^{\pi} V(x)\,dx + \sum_{k=2}^{N} \gamma_k n^{-2k-1} + O(n^{-2N-3}) \tag{1.2.20}$$

for any $N \in \mathbb{N}$, such that the following properties hold true:

$$\frac{1}{h_n}\theta(\pi,h_n) - \frac{1}{h_n}\theta(-\pi,h_n) - 2\pi n = O(n^{-\infty}) \tag{1.2.21}$$

$$\begin{gathered}\theta'(x,0) \equiv 1, |(\partial_x^\alpha\partial_h^\beta\theta')(-\pi,h) - (\partial_x^\alpha\partial_h^\beta\theta')(\pi,h)| = O(h^\infty),\\ |\partial_x^\alpha b(-\pi,h) - \partial_x^\alpha b(\pi,h)| = O(h^\infty)\end{gathered} \qquad \forall\alpha,\beta\in\mathbb{N}, \tag{1.2.22}$$

and such that if one sets

$$\underline{U}_n(x) = e^{i\theta(x,h_n)/h_n} b(x,h_n) \tag{1.2.23}$$

conditions (i) and (iii) of the statement of proposition 1.2.3 hold true.

Proof: We look for a formal series in h, $\Phi(x,h)$, with smooth coefficients in $x \in [-\pi,\pi]$, such that $\mathrm{Im}\,\Phi(x,0) \equiv 0$, and the semi-classical equation

$$(-h^2\partial_x^2 + h^2 V(x) - 1)e^{i\Phi(x,h)/h} = 0 \tag{1.2.24}$$

be satisfied formally. We get, denoting by Φ', Φ'' x-derivatives, the formal equation

$$\Phi'(x,h)^2 - 1 - ih\Phi''(x,h) + h^2 V(x) \equiv 0. \tag{1.2.25}$$

We look for a solution $\Phi'(x,h) = \sum_{k=0}^{+\infty} h^k \Phi'_k(x)$ with $\Phi'_0 \equiv 1$, Φ'_{2k} real, Φ'_{2k+1} purely imaginary. Identifying powers of h we get for $k \geq 1$,

$$\Phi'_k(x) = -\frac{1}{2}V(x)\delta_{k2} - \frac{1}{2}\sum_{\ell=1}^{k-1} \Phi'_\ell(x)\Phi'_{k-\ell}(x) + \frac{i}{2}\Phi''_{k-1}(x)$$

whence

$$\Phi'_1(x) \equiv 0,\ \Phi'_2(x) = -\frac{1}{2}V(x),\ \Phi'_k(x)\ 2\pi\text{-periodic for any } k. \tag{1.2.26}$$

Taking the imaginary part of (1.2.25), we get

$$\operatorname{Re}\Phi'(x,h)\operatorname{Im}\Phi'(x,h) = \frac{h}{2}\operatorname{Re}\Phi''(x,h).$$

We choose for the equation on $\operatorname{Im}\Phi$ the solution

$$\operatorname{Im}\Phi(x,h) = \frac{h}{2}\log[\operatorname{Re}\Phi'(x,h)], \tag{1.2.27}$$

where the right hand side is well defined since $\operatorname{Re}\Phi'(x,0) \equiv 1$. We thus see that $\operatorname{Im}\Phi(x,h)$ is 2π-periodic in x and odd in h. We may write using (1.2.26)

$$\Phi(\pi,h) - \Phi(-\pi,h) = \int_{-\pi}^{\pi} \operatorname{Re}\Phi'(x,h)\,dx = 2\pi - \frac{h^2}{2}\int_{-\pi}^{\pi} V(x)\,dx + \sum_{k=2}^{+\infty} A_k h^{2k} \tag{1.2.28}$$

for some real constants A_k. Then $e^{i\Phi(x,h)/h}$ will be 2π-periodic if and only if there is $n \in \mathbb{N}$ with $\Phi(\pi,h) - \Phi(-\pi,h) = 2\pi nh$. By (1.2.28), the h-solutions of this equation for n large enough form a sequence $(h_n)_n$ of $\mathbb{R}_+^*$, converging to zero, and having asymptotic expansion

$$h_n = \frac{1}{n} - \frac{1}{4\pi n^3}\int_{-\pi}^{\pi} V(x)\,dx + \cdots$$

Comparison with (1.2.1) shows that $h_n^{-1} - \omega(n) = O(n^{-3})$.

We denote by $\theta(x,h)$ (resp. $\tilde{b}(x,h)$) a smooth function of (x,h) on $[-\pi,\pi]\times[0,\delta_0]$, even in h, whose difference with $\operatorname{Re}\Phi(x,h)$ (resp. $e^{-\operatorname{Im}\Phi(x,h)/h}$) is tangent to 0 at infinite order, as well as its derivatives, when $h \to 0$, uniformly in $x \in [-\pi,\pi]$. Since $\operatorname{Im}\Phi(x,h)$ and $\operatorname{Re}\Phi'(x,h)$ are 2π-periodic for any h, (1.2.22) with b replaced by $\tilde{b}$ holds true. Moreover, by (1.2.26), (1.2.27), $\tilde{b}(x,h) = 1 + O(h^2)$ uniformly in $x \in [-\pi,\pi]$, so $\|\tilde{b}(\cdot,h)\|_{L^2([-\pi,\pi])} = \sqrt{2\pi} + O(h^2)$. If we set $b(x,h) = \tilde{b}(x,h)/\|\tilde{b}(\cdot,h)\|_{L^2}$, we thus obtain a function satisfying the last relation (1.2.22). The equality (1.2.21) follows from the definition of h_n. Define now $\underline{U}_n(x,h) = e^{i\theta(x,h_n)/h_n} b(x,h_n)$. It obeys the properties of (i) of proposition 1.2.3. Moreover, by (1.2.24), we have the equality $(-\Delta + V - h_n^{-2})\underline{U}_n = O(h_n^\infty)$ on $[-\pi,\pi]$. If U_n is the 2π-periodization of $\underline{U}_n$, then U_n is in $L^2(\mathbb{S}^1,\mathbb{C})$, but not in $C^\infty(\mathbb{S}^1)$, since it has, as well as its derivatives, jumps of magnitude $O(h_n^\infty)$ at $\pi \mod 2\pi$. Consequently, $(-\Delta + V - h_n^{-2})U_n = \alpha_n\delta_\pi + \beta_n\delta'_\pi + g_n(x)$ where $\alpha_n, \beta_n = O(h_n^\infty)$, g_n is C^∞ on $[-\pi,\pi]$ and $O(h_n^\infty)$. This gives the first inequality in (1.2.19). The second one follows from the fact that by (1.2.23), $\nabla U_n = \alpha_n\delta_\pi + r_n$ with $\alpha_n = O(h_n^\infty)$, $\|r_n\|_{L^2} = O(h_n^{-1})$, whence $\|\nabla U_n\|_{H^{-1/2-\delta}} = O(h_n^{-1})$ for any $\delta > 0$. □

We want now to express the quantities (1.2.16) in terms of Fourier integrals. Remind that we consider a pseudo-differential operator T of order 0, expressed in terms of its symbol a by (1.2.9).

Lemma 1.2.5 *There is $\nu \in \mathbb{R}_+$, a finite set of indices $\mathcal{J}$, and for any $N \in \mathbb{N}$, functions $r_N^\pm : \mathbb{N}_\tau \times \mathbb{N}_\tau \to \mathbb{C}$ satisfying for any α, β, γ*

$$(1.2.29) \qquad |\partial_n^\alpha (\partial_{n'}^*)^\beta (\partial_n - \partial_{n'}^*)^\gamma r_N(n, n')| \leq C_{\alpha\beta\gamma N}(n+n')^{-N-\gamma}|a|_{N+\alpha+\beta+\gamma+\nu}$$

and a family of functions $A_N^{j,\pm} : \mathbb{R}^3 \times \mathbb{R}_+^2 \to \mathbb{C}$,

$$(x, y, \xi, \omega, \omega') \to A_N^{j,\pm}(x, y, \xi, \omega, \omega'),$$

compactly supported relatively to (x, y, ξ), smooth in (ω, ω'), satisfying for $|\omega - \omega'| \leq \frac{1}{2}(\omega + \omega')$ estimates of type

$$(1.2.30) \quad |\partial_\omega^\alpha \partial_{\omega'}^\beta (\partial_\omega + \partial_{\omega'})^\gamma A_N^{j,\pm}(x, y, \xi, \omega, \omega')| \\ \leq C_{\alpha\beta\gamma N N'}|a|_{N+N'+\alpha+\beta+\gamma}(1 + |x-y|\omega)^{-N'}\langle \omega \pm \omega' \rangle^{-N}(\omega + \omega')^{-\gamma}$$

for any $\alpha, \beta, \gamma, N'$, such that if

$$(1.2.31) \qquad J_N^{j,\pm}(\omega, \omega') = \omega \int_{\mathbb{R}^3} e^{i[\omega(x-y)\xi + \omega\theta(y, \frac{1}{\omega}) \pm \omega'\theta(x, \frac{1}{\omega'})]} A_N^{j,\pm}(x, y, \xi, \omega, \omega')\, dx dy d\xi,$$

one has for $|n - n'| \leq \frac{1}{2}(n + n')$

$$(1.2.32) \qquad I_\pm(n, n') = \sum_{j \in \mathcal{J}} J_N^{j,\pm}(h_n^{-1}, h_{n'}^{-1}) + r_N^\pm(n, n').$$

Proof: If we use (1.2.9), (1.2.13) and a partition of unity in y, we may write Tv as the sum of Rv – where R is a smoothing operator whose contribution will be discussed at the end of the proof – and of a finite sum of integrals of form

$$(1.2.33) \qquad \int_{\mathbb{R}^2} e^{i(x-y)\xi} \tilde{a}(x, y, \xi) v(y)\, dy d\xi$$

where v is the 2π-periodic extension of $v \in L^2(\mathbb{S}^1, \mathbb{R})$, where $\tilde{a}$ is C^∞ in (x, y, ξ), compactly supported in (x, y), and satisfies

$$(1.2.34) \qquad |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \tilde{a}(x, y, \xi)| \leq C_{\alpha\beta\gamma}(1 + |\xi|)^{-\gamma}$$

with constants $C_{\alpha\beta\gamma}$ controlled in terms of $|a|_{\alpha+\beta+\gamma}$. Let $\chi_1 \in C^\infty(\mathbb{R})$, $\chi_1 \equiv 0$ on $[-1, 1]$, $\chi_1 \equiv 1$ outside $[-2, 2]$, and define

$$(1.2.35) \qquad T^n v(x) = \int e^{i(x-y)\xi} \tilde{a}(x, y, \xi) \chi_1(n^{-2}\xi) v(y)\, dy d\xi.$$

Let us take $v = U_n$, 2π-periodic extension of the function $\underline{U}_n$ defined on $[-\pi, \pi]$ by (1.2.23). Remind that U_n is smooth outside $\pi + 2\pi\mathbb{Z}$, and that at all points of $\pi + 2\pi\mathbb{Z}$, U_n as well as its derivatives, have a jump of magnitude $O(n^{-\infty})$. Consequently, when we perform in (1.2.35) one integration by parts in y, we get

$$T^n v(x) = \int e^{i(x-y)\xi} \mathbf{1}_{\{y - \pi \notin 2\pi\mathbb{Z}\}} \partial_y [\tilde{a}(x, y, \xi) \frac{\chi_1(n^{-2}\xi)}{i\xi} U_n(y)]\, dy d\xi + T_1^n w$$

where T_1^n is an operator of order -1, acting on a distribution w which is a finite sum of Dirac masses with coefficients $O(n^{-\infty})$. In particular, $\|T_1^n w\|_{L^2} = O(n^{-\infty})$. If we perform more integrations by parts, we may write, remarking that each integration gains n^{-2} and looses one ∂_y derivative

$$\|T^n v\|_{L^2} \leq C_N |a|_{N+\nu} n^{-2N} \|\underline{U}_n\|_{H^N([-\pi,\pi])}$$

for a fixed $\nu \in \mathbb{R}_+$. Since by (1.2.23), $\|\underline{U}_n\|_{H^N} = O(n^N)$, we see that the contribution of T^n to $I_\pm(n,n')$ contributes to the last term in (1.2.32). This shows that we may, from now on, replace T by the operator T_n defined by

$$T_n v(x) = \int e^{i(x-y)\xi} \tilde{a}(x,y,\xi)\chi(n^{-2}\xi)v(y)\, dyd\xi$$

where $\chi = 1 - \chi_1$, and study instead of $I_-(n,n')$ (resp. $I_+(n,n')$) the quantity $\langle T_n U_n, U_{n'}\rangle$ (resp. $\langle T_n U_n, \overline{U}_{n'}\rangle$) i.e. respectively

$$\int_{\mathbb{R}^3} e^{i(x-y)\xi + \frac{i}{h_n}\theta(y,h_n) \mp \frac{i}{h_{n'}}\theta(x,h_{n'})} \tilde{a}(x,y,\xi)\chi(n^{-2}\xi) b(y,h_n) b^{\mp}(x,h_{n'})\, dxdyd\xi \tag{1.2.36}$$

with $b^+ \equiv b, b^- \equiv \bar{b}$. If we make in (1.2.36) integrations by parts in x or y, because θ or b have jumps at $\pi + 2\pi\mathbb{Z}$, we shall get boundary terms. But (1.2.21), (1.2.22), and the fact that ξ is localized in a region where $|\xi| \leq Cn^2$, show us that these contributions will give rise to admissible remainders of type (1.2.29). Consequently, we may argue like if θ and b were C^∞ 2π-periodic functions. Remark that by the first relation (1.2.22), we shall have $|\xi - \frac{1}{h_n}\theta'(y,h_n)| \geq \frac{c}{h_n}$ if h_n is small enough, and either $|\xi| \geq Ah_n^{-1}$ or $|\xi| \leq A^{-1}h_n^{-1}$ for a large enough constant $A > 0$. Consequently, using y-integrations by parts, we see that up to admissible remainders of type (1.2.29), we may in (1.2.36) replace the cut-off $\chi(n^{-2}\xi)$ by $\varphi(h_n\xi)$ with $\varphi \in C_0^\infty(\mathbb{R} - \{0\})$. We are thus reduced to

$$\frac{1}{h_n}\int e^{i\left[\frac{1}{h_n}(x-y)\xi + \frac{1}{h_n}\theta(y,h_n) \mp \frac{1}{h_{n'}}\theta(x,h_{n'})\right]} \tilde{a}\big(x,y,\frac{\xi}{h_n}\big)\varphi(\xi) b(y,h_n) b^{\mp}(x,h_{n'})\, dxdyd\xi. \tag{1.2.37}$$

Define the vector field

$$\begin{aligned} L_\mp(x,y,\omega,\omega',\partial_x+\partial_y) = &\Big(1 + \Big(\omega\theta'\big(y,\frac{1}{\omega}\big) \mp \omega'\theta'\big(x,\frac{1}{\omega'}\big)\Big)^2\Big)^{-1} \\ &\times\Big[1 + \Big(\omega\theta'\big(y,\frac{1}{\omega}\big) \mp \omega'\theta'\big(x,\frac{1}{\omega'}\big)\Big)(\partial_x+\partial_y)\Big]. \end{aligned} \tag{1.2.38}$$

Since $\theta'(x,h)$ is even in h, and $\theta'(x,0) \equiv 1$, we may write

$$\omega\theta'\big(y,\frac{1}{\omega}\big) \mp \omega'\theta'\big(x,\frac{1}{\omega'}\big) = \omega \mp \omega' + \sigma(y,\omega) \mp \sigma(x,\omega') \tag{1.2.39}$$

where $\sigma(y,\omega)$ satisfies for any $\alpha,\gamma \in \mathbb{N}$ (using (1.2.22))

$$\begin{aligned} &|\partial_y^\alpha \partial_\omega^\gamma \sigma(y,\omega)| \leq C_{\alpha\gamma}(1+\omega)^{-1-\gamma} \quad \forall y \in \mathbb{R} - \{\pi + 2\pi\mathbb{Z}\},\ \forall \omega \in \mathbb{R}_+ \\ &[\partial_y^\alpha \partial_\omega^\gamma \sigma] = O(\omega^{-\infty}), \end{aligned}$$

denoting by $[\cdot]$ the jump at $\pi + 2\pi\mathbb{Z}$. Consequently, the coefficients $c(x,y,\omega,\omega')$ of $L_\mp$ satisfy for x,y outside $\pi + 2\pi\mathbb{Z}$,

$$|\partial_x^\delta \partial_y^{\delta'} \partial_\omega^\alpha \partial_{\omega'}^\beta (\partial_\omega + \partial_{\omega'})^\gamma c(x,y,\omega,\omega')| \leq C(1+\omega+\omega')^{-\gamma}\langle \omega \mp \omega'\rangle^{-1} \tag{1.2.40}$$

when $|\omega-\omega'|\le\frac{1}{2}(\omega+\omega')$, with jump conditions

$$[\partial_x^\delta\partial_y^{\delta'}\partial_\omega^\alpha\partial_{\omega'}^\beta c]=O((\omega+\omega')^{-\infty}). \tag{1.2.41}$$

We make in (1.2.37) integrations by parts using the vector field (1.2.38). Again, because of (1.2.41) and (1.2.21), (1.2.22), boundary terms coming from the jumps give rise to remainders of type (1.2.29), and up to such perturbations, we may rewrite (1.2.37) as

$$\frac{1}{h_n}\int e^{i\left[\frac{1}{h_n}(x-y)\xi+\frac{1}{h_n}\theta(y,h_n)\mp\frac{1}{h_{n'}}\theta(x,h_{n'})\right]}({}^tL_\mp)^N\left[\tilde a\big(x,y,\frac{\xi}{h_n}\big)\varphi(\xi)b(y,h_n)b^\mp(x,h_{n'})\right]dxdyd\xi. \tag{1.2.42}$$

If $L_0(x-y,\omega,\partial_\xi)=(1+\omega^2(x-y)^2)^{-1}(1+\omega(x-y)\cdot\partial_\xi)$, the coefficients of L_0 satisfy estimates

$$|\partial_\omega^\alpha c(x-y,\omega)|\le C_\alpha(1+\omega|x-y|)^{-1}\omega^{-\alpha}. \tag{1.2.43}$$

Integrating by parts using L_0, we obtain that (1.2.42) may be written as $J_N^\mp(h_n^{-1},h_{n'}^{-1})$ with

$$J_N^\mp(\omega,\omega')=\omega\int e^{i[\omega(x-y)\xi+\omega\theta(y,\frac{1}{\omega})\mp\omega'\theta(x,\frac{1}{\omega'})]}A_N^\mp(x,y,\xi,\omega,\omega')\,dxdyd\xi$$

with

$$A_N^\mp=({}^tL_0)^{N'}({}^tL_\mp)^N\left[\tilde a(x,y,\omega\xi)\varphi(\xi)b\big(y,\frac{1}{\omega}\big)b^\mp\big(x,\frac{1}{\omega'}\big)\right].$$

By (1.2.40), (1.2.43), and (1.2.34), $A_N^\mp$ satisfies (1.2.30). Finally, the contributions $\langle RU_n,U_{n'}\rangle$, $\langle RU_n,\bar U_{n'}\rangle$ of the smoothing operator in (1.2.13) to I_+,I_- contribute to $r_N^\pm$ in (1.2.32), using (1.2.23) and integrations by parts. This proves the lemma. □

Proof of proposition 1.2.3: By lemma 1.2.4, conditions (i) and (iii) of the statement of the proposition hold true. Let us prove (1.2.18). Since $h_n^{-1}=n+O(1/n)$, if we plug (1.2.30) with $\alpha=\beta=\gamma=0$ inside (1.2.31) and integrate in y, we get from (1.2.32) that there is a fixed $\nu\in\mathbb{R}_+$ such that for any N, $|I_+(n,n')|\le C_N(n+n')^{-N}|a|_{N+\nu}$ when $|n-n'|\le\frac{1}{2}(n+n')$. This implies (1.2.18).

To show (1.2.17), let us prove first that for $|\omega-\omega'|\le\frac{1}{2}(\omega+\omega')$

$$|\partial_\omega^\alpha\partial_{\omega'}^\beta(\partial_\omega+\partial_{\omega'})^\gamma J_N^{j,-}(\omega,\omega')|\le C\langle\omega-\omega'\rangle^{-N}(\omega+\omega')^{-\gamma}|a|_{\alpha+\beta+\gamma+N+2}. \tag{1.2.44}$$

Remark first that if we make act $\partial_\omega+\partial_{\omega'}$ on the phase of $J_N^{j,-}$, we get either a contribution which is $O(\omega^{-1}+\omega'^{-1})$, or a quantity like $i(x-y)\xi$ or $i\big[\theta(y,\frac{1}{\omega})-\theta(x,\frac{1}{\omega'})\big]$, in which, modulo a $O(\omega^{-1}+\omega'^{-1})$ term, we may factor out $x-y$. The decay given by the N' exponent in (1.2.30) allows one to transform such a term in a gain of one negative power of ω. Consequently, (1.2.44) follows from y-integrations of estimates (1.2.30). We have then to show that (1.2.44) implies that

$$\partial_n^\alpha(\partial_{n'}^*)^\beta(\partial_n-\partial_{n'}^*)^\gamma J_N^{j,-}\Big(\frac{1}{h_n},\frac{1}{h_{n'}}\Big) \tag{1.2.45}$$

is estimated by the right hand side of (1.2.17). Call $\tilde\omega(\lambda)$ a symbol of order 1 defined on $\mathbb{R}_+$, such that according to (1.2.20), $h_n^{-1}-\tilde\omega(n)=O(n^{-\infty})$. Up to terms verifying estimates of type (1.2.29) we may, instead of (1.2.45), bound

$$\partial_n^\alpha(\partial_{n'}^*)^\beta(\partial_n-\partial_{n'}^*)^\gamma J_N^{j,-}(\tilde\omega(n),\tilde\omega(n')).$$

We use induction on $\alpha+\beta+\gamma$: set for $t \in [0,1]$, $\Omega(n,t) = t\tilde{\omega}(n+1) + (1-t)\tilde{\omega}(n)$ so that

$$\begin{aligned}(\partial_n - \partial^*_{n'})J^{j,-}_N(\tilde{\omega}(n),\tilde{\omega}(n')) &= J^{j,-}_N(\tilde{\omega}(n+1),\tilde{\omega}(n')) - J^{j,-}_N(\tilde{\omega}(n),\tilde{\omega}(n'-1))\\ &= \int_0^1 (\partial_\omega J^{j,-}_N)(\Omega(n,t),\Omega(n'-1,t))\,dt(\tilde{\omega}(n+1)-\tilde{\omega}(n))\\ &\quad + \int_0^1 (\partial_{\omega'} J^{j,-}_N)(\Omega(n,t),\Omega(n'-1,t))\,dt(\tilde{\omega}(n')-\tilde{\omega}(n'-1)).\end{aligned}$$

Since $\tilde{\omega}(\lambda)-\lambda$ is a symbol of order -1, we may write this as

$$\begin{aligned}\int_0^1 (\partial_\omega + \partial_{\omega'})J^{j,-}_N(\Omega(n,t),\Omega(n'-1,t))\,dt + \int_0^1 \partial_\omega J^{j,-}_N(\Omega(n,t),\Omega(n'-1,t))\,dt\tilde{\omega}_{-2}(n)\\ + \int_0^1 \partial_{\omega'} J^{j,-}_N(\Omega(n,t),\Omega(n'-1,t))\,dt\tilde{\omega}_{-2}(n'-1)\end{aligned}$$

for a new symbol of order -2, $\tilde{\omega}_{-2}(\lambda)$. This shows that we gained one (actually two) negative powers of $n+n'$ in the last two integrals – when $|n-n'| \le \frac{1}{2}(n+n')$ –, and also one such power in the first one, because of (1.2.44). Moreover, $\Omega(n,t)$ satisfies the same assumptions as $\tilde{\omega}(n)$, which allows one to proceed with the induction. This concludes the proof of the proposition. □

Lemma 1.2.6 *Let $\lambda \to \omega(\lambda)$ be the symbol defined after (1.2.1). Then*

$$\frac{1}{h_n} - \omega(n) = O(n^{-\infty}). \tag{1.2.46}$$

Moreover, for n large enough, there is a real valued orthonormal basis $(\varphi^1_n, \varphi^2_n)$ of the space E_n such that

$$\Big\|\varphi^1_n - \frac{U_n+\bar{U}_n}{\sqrt{2}}\Big\|_{L^2} = O(n^{-\infty}), \quad \Big\|\varphi^2_n - \frac{U_n-\bar{U}_n}{i\sqrt{2}}\Big\|_{L^2} = O(n^{-\infty}). \tag{1.2.47}$$

Proof: We denote by F_n the span of (U^1_n, U^2_n) in $L^2(\mathbb{S}^1,\mathbb{R})$, where $U^1_n = \frac{U_n+\bar{U}_n}{\sqrt{2}}, U^2_n = \frac{U_n-\bar{U}_n}{i\sqrt{2}}$. Then for $v \in F_n$, if $P = -\frac{d^2}{dx^2}+V(x)$, we have by (iii) of proposition 1.2.3

$$\|(P-h_n^{-2})v\|_{H^{-2}} = O(n^{-\infty}) \tag{1.2.48}$$

uniformly for v staying in the unit ball of F_n. In the same way, since E_n is the range of the spectral projector Π_n associated to the couple of eigenvalues $(\omega_n^-)^2 \le (\omega_n^+)^2$, we have by (1.2.2)

$$\|(P-\omega(n)^2)v\|_{H^{-2}} = O(n^{-\infty}) \tag{1.2.49}$$

uniformly for v in the unit ball of E_n (actually, the above relation holds true even for the L^2 norm). We shall denote by $E_n^\perp$ the orthogonal complement of E_n in H^{-2}, by $\Pi_n^\perp : H^{-2} \to E_n^\perp$ the orthogonal projection, and shall also use the notation Π_n for the orthogonal projector from H^{-2} to E_n. We set $Q_n = \Pi_n^\perp(P-\omega(n)^2\mathrm{Id})\Pi_n^\perp$ considered as a bounded operator from $E_n^\perp \cap L^2$

to $E_n^\perp$. Since the eigenvalues of P different from $(\omega_n^+)^2$ and $(\omega_n^-)^2$ lie at a distance from $\omega(n)^2$ bounded from below by a fixed constant, Q_n is invertible, with inverse $Q_n^{-1} : E_n^\perp \to E_n^\perp \cap L^2$ whose norm in $\mathcal{L}(H^{-2}, L^2)$ depends on n, but with $\|Q_n^{-1}\|_{\mathcal{L}(H^{-2},H^{-2})}$ uniformly bounded. Since we have seen in proposition 1.2.3 that $\omega(n) - h_n^{-1} = O(n^{-3})$, the operator

$$\mathrm{Id} - Q_n^{-1}(h_n^{-2} - \omega(n)^2) \tag{1.2.50}$$

will be invertible, as an operator from $E_n^\perp$ to $E_n^\perp$ endowed with the H^{-2} norm, for large enough n. If v is in the unit ball of L^2, we have

$$Q_n v = \Pi_n^\perp(P - \omega(n)^2\mathrm{Id})\Pi_n^\perp v = \Pi_n^\perp(P - \omega(n)^2\mathrm{Id})v - \Pi_n^\perp(P - \omega(n)^2\mathrm{Id})\Pi_n v. \tag{1.2.51}$$

By (1.2.2), the last term has L^2 (or H^{-2}) norm $O(n^{-\infty})$. If we assume moreover that $v \in F_n$, and write

$$(P - \omega(n)^2\mathrm{Id})v = (h_n^{-2} - \omega(n)^2)v + (P - h_n^{-2})v,$$

the last term has H^{-2} norm $O(n^{-\infty})$ by (1.2.48). We deduce from this equality and (1.2.51)

$$(Q_n - (h_n^{-2} - \omega(n)^2)\mathrm{Id})\Pi_n^\perp v = r_n$$

with $r_n \in E_n^\perp$, $\|r_n\|_{H^{-2}} = O(n^{-\infty})$. We deduce from the invertibility of Q_n and of (1.2.50) for large enough n that

$$\|\Pi_n^\perp v\|_{H^{-2}} = O(n^{-\infty}). \tag{1.2.52}$$

We set for n large enough $\psi_n^1 = \Pi_n U_n^1, \psi_n^2 = \Pi_n U_n^2$. The above equality implies

$$\|\psi_n^1 - U_n^1\|_{H^{-2}} = O(n^{-\infty}), \quad \|\psi_n^2 - U_n^2\|_{H^{-2}} = O(n^{-\infty}). \tag{1.2.53}$$

Moreover, since ψ_n^j is in the range of Π_n, $\|\psi_n^j\|_{H^{\frac{1}{2}-\delta}} \le Ch_n^{-\frac{1}{2}+\delta}$ for any $\delta > 0$, so that using (1.2.19) $\|\psi_n^j - U_n^j\|_{H^{\frac{1}{2}-\delta}} \le Ch_n^{-1}$. Interpolating with (1.2.53), we get

$$\|\psi_n^j - U_n^j\|_{L^2} = O(n^{-\infty}) \quad j = 1, 2. \tag{1.2.54}$$

Since $\|U_n\|_{L^2} = 1$, and $\langle U_n, \bar{U}_n\rangle = O(n^{-\infty})$ by (1.2.16) and (1.2.18), we deduce from (1.2.54) and the definition of U_n^1, U_n^2

$$\langle \psi_n^1, \psi_n^2\rangle = O(n^{-\infty}), \quad \|\psi_n^j\|_{L^2}^2 - 1 = O(n^{-\infty}). \tag{1.2.55}$$

We define now $(\varphi_n^1, \varphi_n^2)$ as a Gram-Schmidt orthonormalization of (ψ_n^1, ψ_n^2). Then (1.2.47) follows from (1.2.54), (1.2.55). To show (1.2.46), we take $v \in F_n$ of norm 1. We write

$$(\omega(n)^2 - h_n^{-2})\Pi_n v = -(P - \omega(n)^2)\Pi_n v + (P - h_n^{-2})v - P\Pi_n^\perp v + h_n^{-2}\Pi_n^\perp v.$$

By (1.2.48), (1.2.49) the H^{-2} norm of the first two terms in the right hand side is $O(n^{-\infty})$. By (1.2.52), the H^{-4} norm of the last two terms is $O(n^{-\infty})$. Consequently

$$(\omega(n)^2 - h_n^{-2})\|\Pi_n v\|_{H^{-4}} = O(n^{-\infty}).$$

To get (1.2.46) and conclude the proof, we just need to see that $\|\Pi_n v\|_{H^{-4}} \sim n^{-4}\|\Pi_n v\|_{L^2} \geq cn^{-4}$. We have, since v is in the unit ball of F_n, $\|\Pi_n^{\perp} v\|_{H^1} \leq C\|v\|_{H^1} \leq Cn$. Interpolating with (1.2.52), we get $\|\Pi_n^{\perp} v\|_{L^2} = O(n^{-\infty})$, whence the wanted lower bound, $\|\Pi_n v\|_{L^2} \geq c$. □

Proof of theorem 1.2.1: For n large enough, we take for $(\varphi_n^1, \varphi_n^2)$ the orthonormal basis of E_n given by lemma 1.2.6. For small values of n, we take any orthonormal basis of E_n. Remark first that if $|n - n'| \geq c(n + n')$ for some $c > 0$, estimate (1.2.14) holds true. Actually, one has a general estimate

$$|\langle \Pi_n u, T\Pi_{n'} v\rangle| \leq C_N \langle n - n'\rangle^{-N} |a|_{\nu+N} \|u\|_{L^2} \|v\|_{L^2}$$

for a fixed $\nu \in \mathbb{R}_+$ (see for instance [10], proposition 1.2.2 and lemma 1.2.3). This implies that if $|n - n'| \geq c(n+n')$, $|\langle \varphi_n^j, T\varphi_{n'}^{j'}\rangle|$ is bounded from above by $C_N(n+n')^{-N}|a|_{\nu+N}$, which is better than the wanted estimate (1.2.14). We may thus assume $|n - n'| \leq c(n + n')$ and n, n' large enough. Then using (1.2.47) we get that up to $O((n+n')^{-\infty})$ terms, $\langle \varphi_n^j, T\varphi_{n'}^{j'}\rangle$ may be written as linear combinations of $I_-(n, n')$ and $I_+(n, n')$. Formulas (1.2.17), (1.2.18) of proposition 1.2.3 give then (1.2.14). This concludes the proof of the theorem. □

2 Paradifferential symbolic calculus

The aim of this section is to develop a symbolic calculus, analogous to Bony's paradifferential calculus [4], for symbols defined on a discrete set instead of an open subset of the euclidean space. As will be clear in section 4, we shall need such an extension, as the symbols which will naturally appear in reductions of the quasi-linear equation (1.1.4) will be defined on $\mathbb{N}^p$, and will not have any nice extension to $\mathbb{R}^p$.

2.1 Symbols and quantization

We first fix some notations. We shall consider G a finite dimensional real vector space, and assume given an orthonormal decomposition

$$L^2(\mathbb{S}^1, G) = \bigoplus_{k \geq \tau} E_k \tag{2.1.1}$$

where E_k is a finite dimensional subspace of dimension $K(k)$ and $\tau \in \mathbb{N}^*$. We assume $K(k)$ independent of k for k large enough, and denote by K this value. We assume that each E_k is endowed with a nice orthonormal basis $(\varphi_k^j)_{1\leq j \leq K(k)}$ i.e. an orthonormal basis such that, for any k, k', for given pseudo-differential operators T_1, T_2 of order 0, for any function $a \in C^\infty(\mathbb{S}^1, \mathbb{R})$, we have estimates of type (1.2.15)

$$|\partial_k^\alpha (\partial_{k'}^*)^\beta (\partial_k - \partial_{k'}^*)^\gamma \langle T_1 \varphi_k^j, a(x) T_2 \varphi_{k'}^{j'}\rangle| \leq C\langle k - k'\rangle^{-N} (k + k')^{-\gamma} \sum_{\ell=0}^{\alpha+\beta+\gamma+N+\nu} \|\partial^\ell a\|_{L^\infty}, \tag{2.1.2}$$

where $1 \leq j \leq K(k)$, $1 \leq j' \leq K(k')$ and ν is a fixed positive constant. We shall denote by $\mathcal{E}$ the algebraic direct sum of the E_k's, and will use $\mathcal{E}$ as a space of test functions.

If $n = (n_0, \dots, n_{p+1}) \in \mathbb{N}_\tau^{p+2}$ we define

$$n' = (n_1, \dots, n_p), |n'| = \max(n_1, \dots, n_p). \tag{2.1.3}$$

Moreover, if n_i is such that $n_i = \max(n_0, \dots, n_{p+1})$ we set

$$\max{}_2(n_0, \dots, n_{p+1}) = \max(\{n_0, \dots, n_{p+1}\} - \{n_i\}) \tag{2.1.4}$$

and if $n_j, j \neq i$, is such that $n_j = \max_2(n_0, \dots, n_{p+1})$ we define

$$\begin{aligned} \mu(n_0, \dots, n_{p+1}) &= \max(\{n_0, \dots, n_{p+1}\} - \{n_i, n_j\}) \\ S(n_0, \dots, n_{p+1}) &= |n_i - n_j| + \mu(n_0, \dots, n_{p+1}). \end{aligned} \tag{2.1.5}$$

By convention, we set $\max_2 n_0 = 1$, $\mu(n_0, n_1) = 1$. We denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$ and by Π_k the orthogonal projector from $L^2(\mathbb{S}^1, G \otimes \mathbb{K})$ to $E_k \otimes \mathbb{K}$ and set

$$\begin{aligned} \mathcal{F}_k :& L^2(\mathbb{S}^1, G \otimes \mathbb{K}) \longrightarrow \mathbb{K}^{K(k)} \\ & u \to (\langle u, \varphi_k^j \rangle)_{1 \leq j \leq K(k)}. \end{aligned} \tag{2.1.6}$$

Then $\mathcal{F}_k$ is an isometry when restricted to $E_k \otimes \mathbb{K}$, if we endow $\mathbb{K}^{K(k)}$ with the ℓ^2 norm. We denote by $\mathcal{F}_k^*$ the adjoint of $\mathcal{F}_k$ from $(\mathbb{K}^{K(k)})^* \simeq \mathbb{K}^{K(k)}$ to $(L^2)' \simeq L^2$. We have for $V = (V_j)_{1 \leq j \leq K(k)} \in \mathbb{K}^{K(k)}$

$$\mathcal{F}_k^* V = \sum_{j=1}^{K(k)} V_j \varphi_k^j(x) \tag{2.1.7}$$

and the relations

$$\mathcal{F}_k^* = \Pi_k \circ \mathcal{F}_k^*,\ \Pi_k = \mathcal{F}_k^* \circ \mathcal{F}_k,\ \mathcal{F}_k \circ \mathcal{F}_k^* = \mathrm{Id}_{\mathbb{K}^{K(k)}},\ \mathcal{F}_k = \mathcal{F}_k \circ \Pi_k. \tag{2.1.8}$$

If $U = (u_1, \dots, u_p) \in (L^2)^p$ and $n' = (n_1, \dots, n_p) \in \mathbb{N}_\tau^p$ we denote

$$\Pi_{n'} U = (\Pi_{n_1} u_1, \dots, \Pi_{n_p} u_p). \tag{2.1.9}$$

We shall always denote by $\|\cdot\|$ the $\mathcal{L}(\ell^2, \ell^2)$ norm of linear maps between euclidean spaces (or the corresponding norm of matrices). Let us define the first class of symbols we shall use.

Definition 2.1.1 *Let $d \in \mathbb{R}, \nu \in \mathbb{R}_+, p \in \mathbb{N}, N_0 \in \mathbb{N}^*$ be given. We denote by $\Sigma_{p,N_0}^{d,\nu}$ the space of maps*

$$\begin{aligned} (u_1, \dots, u_p, n_0, n_{p+1}) &\longrightarrow a(u_1, \dots, u_p; n_0, n_{p+1}) \\ \mathcal{E} \times \cdots \times \mathcal{E} \times \mathbb{N}_\tau \times \mathbb{N}_\tau &\longrightarrow \mathcal{L}(\mathbb{K}^{K(n_{p+1})}, \mathbb{K}^{K(n_0)}) \end{aligned} \tag{2.1.10}$$

such that a is $\mathbb{R}$-p-linear in $(u_1, \dots, u_p)$ and satisfies for some $\delta \in]0, 1[$ conditions:

(i)$_\delta$ For any $U = (u_1, \dots, u_p) \in \mathcal{E}^p$, any $n = (n_0, n', n_{p+1}) \in \mathbb{N}_\tau^{p+2}$ (with $n' = (n_1, \dots, n_p)$), $a(\Pi_{n'}U; n_0, n_{p+1}) \equiv 0$ unless

$$|n'| \leq \delta(n_0 + n_{p+1}) \text{ and } |n_0 - n_{p+1}| \leq \delta(n_0 + n_{p+1}). \tag{2.1.11}$$

(ii) For any $N \in \mathbb{N}$, any $\alpha, \beta, \gamma \in \mathbb{N}$, there is $C > 0$ such that for any $n = (n_0, n', n_{p+1}) \in \mathbb{N}_\tau^{p+2}$ as above, any $U = (u_1, \dots, u_p) \in \mathcal{E}^p$, one has the estimate

$$\begin{aligned} &\|\partial_{n_0}^\alpha (\partial_{n_{p+1}}^*)^\beta (\partial_{n_0} - \partial_{n_{p+1}}^*)^\gamma a(\Pi_{n'}U; n_0, n_{p+1})\| \\ &\qquad\qquad \leq C(n_0 + n_{p+1})^{d-\gamma} \frac{|n'|^{\nu+N+(\alpha+\beta+\gamma)N_0}}{(|n_0 - n_{p+1}| + |n'|)^N} \prod_{j=1}^p \|u_j\|_{L^2}. \end{aligned} \tag{2.1.12}$$

We shall call symbols in the preceding class paradifferential symbols. We may of course extend (2.1.10) to a $\mathbb{C}$-p-linear map defined on $(\mathcal{E} \otimes \mathbb{C}) \times \cdots \times (\mathcal{E} \otimes \mathbb{C}) \times \mathbb{N}_\tau \times \mathbb{N}_\tau$.

Remarks • When we make act $\partial_{n_{p+1}}^*$ several times on $a(\Pi_{n'}U; n_0, n_{p+1})$, we might, for small values of n_{p+1}, have to calculate a at integers smaller than τ. We decide to extend $a(\cdot\,; n_0, n_{p+1})$ as 0 for $n_0 < \tau$ or $n_{p+1} < \tau$.

• When $|n'|$ is bounded, estimate (2.1.12) is similar to the estimate (2.1.2) defining nice basis. When $|n'| \to +\infty$, we have an extra loss of powers of $|n'|$, coming from $\|\partial^\ell a\|_{L^\infty}$ in (2.1.2), and from degenerate ellipticity estimates of some symbols that we shall have to include in our classes.

• When $p = 0$, we set by convention $|n'| = 1$ in the above definition, and in all forthcoming formulas.

Let us quantize the above symbols.

Definition 2.1.2 *For $a \in \Sigma_{p,N_0}^{d,\nu}$ and $U = (u_1, \dots, u_p) \in \mathcal{E}^p$, $u_{p+1} \in \mathcal{E}$, we define*

$$\mathrm{Op}(a(U; \cdot))u_{p+1} = \sum_{n_0 \in \mathbb{N}_\tau} \sum_{n_{p+1} \in \mathbb{N}_\tau} \mathcal{F}_{n_0}^* \big[a(U; n_0, n_{p+1}) \mathcal{F}_{n_{p+1}} u_{p+1}\big]. \tag{2.1.13}$$

Let us explain the origin of the above definition. Assume for instance that each E_k is one dimensional, spanned by a function φ_k. If $a, u \in L^2$, we may write

$$\begin{aligned} au &= \sum_{n_{p+1}} a(x) \langle u, \varphi_{n_{p+1}} \rangle \varphi_{n_{p+1}} \\ &= \sum_{n_0} \sum_{n_{p+1}} \langle a\varphi_{n_{p+1}}, \varphi_{n_0} \rangle \langle u, \varphi_{n_{p+1}} \rangle \varphi_{n_0} \\ &= \sum_{n_0} \sum_{n_{p+1}} \mathcal{F}_{n_0}^* \big[\langle a\varphi_{n_{p+1}}, \varphi_{n_0} \rangle \mathcal{F}_{n_{p+1}} u\big] \end{aligned}$$

using (2.1.6), (2.1.7), and the symbol $\langle a\varphi_{n_{p+1}}, \varphi_{n_0}\rangle$ satisfies by (2.1.2) estimates (2.1.12). Condition (i)$_\delta$ of definition 2.1.1, which is not satisfied in this example, comes from the fact that we want to consider paradifferential operators, instead of pseudo-differential ones.

Let us show that operators of order 0 are bounded on H^s for s large enough.

Proposition 2.1.3 *Let $\nu \in \mathbb{R}_+, N_0 \in \mathbb{N}^*$. There exists $s_0 \in \mathbb{R}$ and for any $s \in \mathbb{R}$, any $d \in \mathbb{R}$, any $p \in \mathbb{N}$, any $a \in \Sigma^{d,\nu}_{p,N_0}$, there is a constant $C > 0$ such that for any $U = (u_1, \dots, u_p) \in \mathcal{E}^p$, any $u_{p+1} \in \mathcal{E}$*

$$\|\mathrm{Op}(a(U;\cdot))u_{p+1}\|_{H^{s-d}} \leq C \prod_{j=1}^{p} \|u_j\|_{H^{s_0}} \|u_{p+1}\|_{H^s}. \tag{2.1.14}$$

In particular, $(U, u_{p+1}) \to \mathrm{Op}(a(U;\cdot))u_{p+1}$ extends as a bounded $(p+1)$-linear map from $(H^{s_0})^p \times H^s$ to H^{s-d}.

Proof: Since $\|v\|^2_{H^s} \sim \sum_n n^{2s}\|\Pi_n v\|^2_{L^2}$, let us estimate $\|\Pi_{n_0}\mathrm{Op}(a(U;\cdot))u_{p+1}\|_{L^2}$. We get using (2.1.12) and condition (i)$_\delta$,

$$\begin{aligned} &n_0^{-d}\Big\|\sum_{n_{p+1}} a(U;n_0,n_{p+1})\mathcal{F}_{n_{p+1}}u_{p+1}\Big\|_{\ell^2} \leq \\ &C\sum_{n_1}\cdots\sum_{n_{p+1}} \frac{|n'|^{\nu+N}}{(|n_0-n_{p+1}|+|n'|)^N} \prod_{j=1}^{p} n_j^{-s_0} n_{p+1}^{-s} c_{n_{p+1}} \prod_{j=1}^{p}\|u_j\|_{H^{s_0}}\|u_{p+1}\|_{H^s} \end{aligned}$$

with $(c_{n_{p+1}})_{n_{p+1}}$ in the unit ball of ℓ^2. Moreover, by condition (i)$_\delta$ of definition 2.1.1, we have $n_{p+1} \sim n_0$ on the summation. Consequently, if we take $N > 1$ and s_0 large enough relatively to ν, we obtain an estimate by $Cn_0^{-s}c'_{n_0}$ for a new ℓ^2-sequence $(c'_{n_0})_{n_0}$, which is the wanted conclusion. □

We shall define now a class of remainder operators.

Definition 2.1.4 *Let $d \in \mathbb{R}, \nu \in \mathbb{R}_+, p \in \mathbb{N}$. We denote by $\mathcal{R}^{d,\nu}_{p+1}$ the space of $(p+1)$-linear maps $M : \mathcal{E} \times \cdots \times \mathcal{E} \to L^2$ such that for any $\ell, N \in \mathbb{N}$, there is $C > 0$ such that for any $(n_0, \dots, n_{p+1}) \in \mathbb{N}^{p+2}_\tau$, any $u_1, \dots, u_{p+1} \in \mathcal{E}$*

$$\begin{aligned} &\|\Pi_{n_0}M(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\|_{L^2} \leq \\ &\qquad Cn_0^d \frac{\max_2(n_1,\dots,n_{p+1})^{\nu+\ell}}{\max(n_1,\dots,n_{p+1})^\ell} \frac{\mu(n_0,\dots,n_{p+1})^N}{S(n_0,\dots,n_{p+1})^N} \prod_{j=1}^{p+1}\|u_j\|_{L^2}. \end{aligned} \tag{2.1.15}$$

Remark that by definition $\mathcal{R}^{d,\nu}_{p+1} \subset \mathcal{R}^{0,\nu+d_+}_{p+1}$, and that M extends to a $\mathbb{C}$-$(p+1)$-linear map defined on $(\mathcal{E}\otimes\mathbb{C})\cdots\times(\mathcal{E}\otimes\mathbb{C})$.

Let us show that up to a remainder operator we always may assume in definition 2.1.1 that condition (i)$_\delta$ is satisfied with an arbitrary small $\delta > 0$.

Lemma 2.1.5 *Let $d \in \mathbb{R}, \nu \in \mathbb{R}_+, p \in \mathbb{N}, N_0 \in \mathbb{N}^*$ be given. There is $\nu' \in \mathbb{R}_+$ such that for any $\delta' \in]0,1[$, any $a \in \Sigma^{d,\nu}_{p,N_0}$, we may find $a_1 \in \Sigma^{d,\nu}_{p,N_0}$, satisfying condition (i)$_{\delta'}$ and $R \in \mathcal{R}^{0,\nu'}_p$, so that for any $U \in \mathcal{E}^p$, $u_{p+1} \in \mathcal{E}$*

$$\mathrm{Op}(a(U;\cdot))u_{p+1} = \mathrm{Op}(a_1(U;\cdot))u_{p+1} + R(U, u_{p+1}).$$

Before starting the proof, let us state a lemma that we shall use several times.

Lemma 2.1.6 *Assume given a family of real valued functions $K_{\alpha\beta\gamma}(\omega,\omega')$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$, such that there are positive constants $C_{\alpha\beta\gamma}$ satisfying*

$$C^{-1}_{\alpha\beta\gamma} K_{\alpha\beta\gamma}(\omega,\omega') \le K_{\alpha\beta\gamma}(\omega+h,\omega'+h') \le C_{\alpha\beta\gamma} K_{\alpha\beta\gamma}(\omega,\omega')$$

for any $\omega,\omega' \in \mathbb{R}^*_+$ large enough, any $(h,h') \in [-1,1]^2$. Let H be a smooth function on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfying for any $\alpha,\beta,\gamma \in \mathbb{N}$, any $\omega,\omega' \in \mathbb{R}^*_+$

$$|\partial^\alpha_\omega \partial^\beta_{\omega'} (\partial_\omega + \partial_{\omega'})^\gamma H(\omega,\omega')| \le K_{\alpha\beta\gamma}(\omega,\omega'). \tag{2.1.16}$$

Then, there are constants $C'_{\alpha\beta\gamma}$ such that for any $\alpha,\beta,\gamma \in \mathbb{N}$, any $n,n' \in \mathbb{N}$ large enough, with $|n-n'| \le \frac{1}{2}(n+n')$

$$|\partial^\alpha_n (\partial^*_{n'})^\beta (\partial_n - \partial^*_{n'})^\gamma H(n,n')| \le C'_{\alpha\beta\gamma} K_{\alpha\beta\gamma}(n,n'). \tag{2.1.17}$$

Proof of lemma 2.1.5: Let χ be a smooth function, with support close enough to 0, equal to one on a neighborhood of zero. Define

$$a_1(U;n_0,n_{p+1}) = \sum_{n'=(n_1,\dots,n_p)} \chi\Big(\frac{n_0-n_{p+1}}{n_0+n_{p+1}}\Big)\chi\Big(\frac{|n'|}{n_0+n_{p+1}}\Big) a(\Pi_{n'}U;n_0,n_{p+1}).$$

Then condition (i)$_{\delta'}$ will be satisfied by a_1 if $\mathrm{Supp}\,\chi$ is small enough. Moreover, using lemma 2.1.6, we see that when $|n_0 - n_{p+1}| \le \frac{1}{2}(n_0+n_{p+1})$

$$\begin{aligned}
&\Big|\partial^\alpha_{n_0}(\partial^*_{n_{p+1}})^\beta(\partial_{n_0}-\partial^*_{n_{p+1}})^\gamma \chi\Big(\frac{n_0-n_{p+1}}{n_0+n_{p+1}}\Big)\Big| \le C_{\alpha\beta\gamma}(n_0+n_{p+1})^{-\gamma}\\
&\Big|\partial^\alpha_{n_0}(\partial^*_{n_{p+1}})^\beta(\partial_{n_0}-\partial^*_{n_{p+1}})^\gamma \chi\Big(\frac{|n'|}{n_0+n_{p+1}}\Big)\Big| \le C_{\alpha\beta\gamma}\frac{|n'|^{\alpha+\beta+\gamma}}{(n_0+n_{p+1})^{\alpha+\beta+\gamma}}.
\end{aligned} \tag{2.1.18}$$

Consequently, using also Leibniz formulas (1.2.6), (1.2.7), we see that estimates (2.1.12) are satisfied by a_1. Finally, since $R = \mathrm{Op}(a-a_1)$,

$$\|\Pi_{n_0} R(\Pi_{n_1}u_1,\dots,\Pi_{n_{p+1}}u_{p+1})\|_{L^2} \le |\!|\!|(a-a_1)(\Pi_{n'}U;n_0,n_{p+1})|\!|\!| \|u_{p+1}\|_{L^2}$$

and since, for the indices to be considered, either $|n'| \ge c(n_0+n_{p+1})$ or $|n_0-n_{p+1}| \ge c(n_0+n_{p+1})$, estimate (2.1.12) gives the upper bound

$$C(n_0+n_{p+1})^{d-N}|n'|^{\nu+N}$$

from which (2.1.15) follows, since

$$\max_2(n_1, \dots, n_{p+1}) \sim |n'|, \mu(n_0, \dots, n_{p+1}) \sim |n'|, S(n_0, \dots, n_{p+1}) \leq C(n_0 + n_{p+1})$$

because of (2.1.11). □

Remainder operators act also on Sobolev spaces:

Lemma 2.1.7 *Let $s_0 > 1$. There is for any $\nu \in \mathbb{R}_+$, any $p \in \mathbb{N}^*, s_1, s_2 \in \mathbb{R}, s_1 + s_2 > \nu + 2$, any $d \in \mathbb{R}$, any $M \in \mathcal{R}^{d,\nu}_{p+1}$, a constant $C > 0$ such that for any $u_1, \dots, u_{p+1} \in \mathcal{E}$, any $n_0 \in \mathbb{N}_\tau$, one has the estimate*

(2.1.19)
$$\|\Pi_{n_0}[M(u_1, \dots, u_{p+1})]\|_{L^2} \leq C n_0^{-s_1-s_2+\nu+2+d} \sum_{1 \leq j_1 \neq j_2 \leq p+1} \|u_{j_1}\|_{H^{s_1}} \|u_{j_2}\|_{H^{s_2}} \prod_{\substack{1 \leq k \leq p+1 \\ k \neq j_1, k \neq j_2}} \|u_k\|_{H^{s_0}}.$$

In particular, M is bounded for any θ from $H^s \times \cdots \times H^s$ to $H^{s+\theta-d}$ if s is large enough with respect to ν and θ and

$$\|M(u, \dots, u)\|_{H^{s+\theta-d}} \leq C\|u\|^{p-1}_{H^{s_0}} \|u\|^2_{H^s}.$$

Proof: We consider the contribution to M of

$$M_1(u_1, \dots, u_{p+1}) = \sum_{n_1 \leq \cdots \leq n_{p+1}} M(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1}).$$

Then by definition 2.1.4

$$\|\Pi_{n_0} M_1(\Pi_{n_1} u_1, \dots, \Pi_{n_{p+1}} u_{p+1})\|_{L^2} \leq C n_0^d \frac{n_p^{\nu+\ell}}{n_{p+1}^\ell} \frac{\mu(n_0, \dots, n_{p+1})^N}{S(n_0, \dots, n_{p+1})^N} \prod_1^{p+1} \|\Pi_{n_j} u_j\|_{L^2}. \tag{2.1.20}$$

For the summation for $n_1 \leq \cdots \leq n_{p+1}$ and $n_p \geq n_0$, we take $\ell = s_1 - \nu, N = 0$. We get the upper bound

$$C n_0^d \sum_{\substack{n_1 \leq \cdots \leq n_{p+1} \\ n_p \geq n_0}} n_{p+1}^{-s_1-s_2+\nu} n_{p-1}^{-s_0} \cdots n_1^{-s_0} \prod_1^{p-1} \|u_j\|_{H^{s_0}} \|u_p\|_{H^{s_1}} \|u_{p+1}\|_{H^{s_2}}$$

which is bounded by the right hand side of (2.1.19) for $s_1 + s_2 > \nu + 2, s_0 > 1$. When we sum for $n_1 \leq \cdots \leq n_{p+1}$ and $n_p < n_0$, we have $\mu(n_0, \dots, n_{p+1}) = n_p$, $S(n_0, \dots, n_{p+1}) = |n_0 - n_{p+1}| + n_p$. We take in (2.1.20) $\ell = s_1 - \nu$, and get

$$C n_0^d \sum_{\substack{n_1 \leq \cdots \leq n_{p+1} \\ n_0 > n_p}} n_1^{-s_0} \cdots n_{p-1}^{-s_0} n_{p+1}^{-s_1-s_2+\nu} n_p^N (|n_0 - n_{p+1}| + n_p)^{-N} \prod_1^{p-1} \|u_j\|_{H^{s_0}} \|u_p\|_{H^{s_1}} \|u_{p+1}\|_{H^{s_2}}.$$

For the sum over $n_{p+1} \geq \frac{1}{2}n_0$, we take $N = 0$ and get the upper bound (2.1.19). For the sum over $n_{p+1} < \frac{1}{2}n_0$, we take $N = s_1 + s_2 - \nu$ and get a bound in terms of

$$\sum_{n_1 \leq \cdots \leq n_{p+1} < \frac{1}{2}n_0} n_1^{-s_0} \cdots n_{p-1}^{-s_0} n_0^{-s_1-s_2+\nu} \leq C n_0^{-s_1-s_2+\nu+2},$$

whence again (2.1.19). □

2.2 Symbolic calculus

We shall prove that the operators we just defined enjoy nice symbolic calculus properties.

Definition 2.2.1 *Let $a \in \Sigma^{d,\nu}_{p,N_0}$. We denote by $a^\bullet$ the symbol defined by*

$$a^\bullet(U; n_0, n_{p+1}) = a(U; n_{p+1}, n_0)^* \tag{2.2.1}$$

where a^* means the adjoint of the operator $a(U; n_{p+1}, n_0)$ acting from $\mathbb{K}^{K(n_0)}$ to $\mathbb{K}^{K(n_{p+1})}$.

Remark that since

$$(\partial_{n_0} - \partial^*_{n_{p+1}})[a^\bullet(U; n_0, n_{p+1})] = [(\partial_X - \partial^*_Y)a(U; X, Y)^*]|_{X=n_{p+1}-1, Y=n_0+1}$$

we get that $a^\bullet \in \Sigma^{d,\nu}_{p,N_0}$. Moreover, it follows from definition 2.1.2 that

$$\mathrm{Op}(a(U; \cdot))^* = \mathrm{Op}(a^\bullet(U; \cdot)),$$

where the star denotes here the adjoint of operators from L^2 to L^2.

Let us study now composition.

Proposition 2.2.2 *(i) Let $\nu \in \mathbb{R}_+, N_0 \in \mathbb{N}^*$. There is $\nu' \in \mathbb{R}_+$ and for any $p, q \in \mathbb{N}, d, d' \in \mathbb{R}$, for any symbols $a \in \Sigma^{d,\nu}_{p,N_0}, b \in \Sigma^{d',\nu}_{q,N_0}$ satisfying condition (i)$_\delta$ of definition 2.1.1 with a small enough $\delta > 0$, there is a symbol $a\#b \in \Sigma^{d+d',\nu'}_{p+q,N_0}$ such that for any $U' = (u_1, \ldots, u_p) \in \mathcal{E}^p$, $U'' = (u_{p+1}, \ldots, u_{p+q}) \in \mathcal{E}^q$, any $u_{p+q+1} \in \mathcal{E}$*

$$\mathrm{Op}(a(U'; \cdot))\mathrm{Op}(b(U''; \cdot))u_{p+q+1} = \mathrm{Op}(a\#b(U', U''; \cdot))u_{p+q+1}. \tag{2.2.2}$$

(ii) Assume moreover that for any U', U'' as above, any large enough $n_0, n_{p+1}, n'_0, n'_{q+1} \in \mathbb{N}_\tau$, the symbols $a(U'; n_0, n_{p+1})$ and $b(U''; n'_0, n'_{q+1})$ commute. Then there is a symbol $c \in \Sigma^{d+d'-1,\nu'}_{p+q,N_0}$ such that

$$[\mathrm{Op}(a(U'; \cdot)), \mathrm{Op}(b(U''; \cdot))]u_{p+q+1} = \mathrm{Op}(c(U', U''; \cdot))u_{p+q+1} \tag{2.2.3}$$

for any $U' \in \mathcal{E}^p, U'' \in \mathcal{E}^q, u_{p+q+1} \in \mathcal{E}$.

Proof: (i) Using definition 2.1.2 and (2.1.8) we get

$$\mathrm{Op}(a(U';\cdot))\mathrm{Op}(b(U'';\cdot))u_{p+q+1} = \sum_{n_0,k,n_{p+q+1}\geq\tau} \mathcal{F}^*_{n_0}[a(U';n_0,k)b(U'';k,n_{p+q+1})\mathcal{F}_{n_{p+q+1}}u_{p+q+1}]$$

and we have to check that

$$(a\#b)(U',U'';n_0,n_{p+q+1}) \stackrel{\text{def}}{=} \sum_{k\geq\tau} a(U';n_0,k)b(U'';k,n_{p+q+1}) \tag{2.2.4}$$

belongs to $\Sigma^{d+d',\nu'}_{p+q,N_0}$ for some ν'. If we set $n' = (n_1,\dots,n_p)$, $n'' = (n_{p+1},\dots,n_{p+q})$ and replace U' (resp. U'') by $\Pi_{n'}U'$ (resp. $\Pi_{n''}U''$) we get from condition (i)$_\delta$ of definition 2.1.1 applied to a, b,

$$\begin{aligned} &|n'| \leq \delta(n_0+k),\ \ |n''| \leq \delta(k+n_{p+q+1})\\ &|n_0-k| \leq \delta(n_0+k),\ \ |k-n_{p+q+1}| \leq \delta(k+n_{p+q+1}) \end{aligned} \tag{2.2.5}$$

which implies that $a\#b$ satisfies (i)$_{4\delta}$ if $\delta>0$ is small enough. One has then to check estimate (2.1.12) for $a\#b$. We shall do that in the proof of (ii) below.

(ii) Before starting the proof, let us gather some formulas that we shall use. Let $c(U;\cdot)$ be a symbol satisfying condition (i)$_\delta$ of definition 2.1.1 with a small enough $\delta>0$. For $h\in\mathbb{Z}$ we have, forgetting the explicit U dependence in the notations, for any $\xi,\eta\in\mathbb{N}$,

$$c(\xi+h,\eta) - c(\xi,\eta-h) = \mathcal{S}((\partial_\xi-\partial^*_\eta)c)(\xi,\eta;h) \tag{2.2.6}$$

where ∂_ξ (resp. ∂^*_η) means derivation with respect to the first (resp. second) argument of $c(\xi,\eta)$, and where

$$\mathcal{S}(c)(\xi,\eta;h) = \sum_{j=0}^{h-1} c(\xi+h-j-1,\eta-j). \tag{2.2.7}$$

We shall denote also

$$\begin{aligned} (\Delta c)(\xi,\eta;k) &= c(\xi,\xi+k) - c(\eta-k,\eta)\\ &= \mathcal{S}((\partial_\xi-\partial^*_\eta)c)(\eta-k,\xi+k;\xi-\eta+k), \end{aligned} \tag{2.2.8}$$

the last equality following from (2.2.6). By direct computation, one checks that

$$\begin{aligned} \partial_\xi[\Delta c(\xi,\eta;k)] &= ((\partial_\xi-\partial^*_\eta)c)(\xi,\xi+k+1)\\ \partial^*_\eta[\Delta c(\xi,\eta;k)] &= ((\partial_\xi-\partial^*_\eta)c)(\eta-k-1,\eta)\\ (\partial_\xi-\partial^*_\eta)[\Delta c(\xi,\eta;k)] &= \Delta((\partial_\xi-\partial^*_\eta)c)(\xi,\eta;k+1) \end{aligned} \tag{2.2.9}$$

and also that

$$\begin{aligned} \partial_\xi\mathcal{S}(c)(\xi,\eta;h) &= \mathcal{S}(\partial_\xi c)(\xi,\eta;h)\\ \partial^*_\eta\mathcal{S}(c)(\xi,\eta;h) &= \mathcal{S}(\partial^*_\eta c)(\xi,\eta;h). \end{aligned} \tag{2.2.10}$$

We consider now the symbol of $[\mathrm{Op}(a(U';\cdot)),\mathrm{Op}(b(U'';\cdot))]$. By (2.2.4), this is equal to the expression $a\#b(U',U'';n_0,n_{p+q+1}) - b\#a(U'',U';n_0,n_{p+q+1})$ i.e.

$$\sum_{k\in\mathbb{N}_\tau} [a(U';n_0,k)b(U'';k,n_{p+q+1}) - b(U'';n_0,k)a(U';k,n_{p+q+1})].$$

Using the assumption $ab = ba$ and changing indexation, we get for large enough n_0, n_{p+q+1}

$$\sum_{k\in\mathbb{Z}}[a(U';n_0,n_0+k)b(U'';n_0+k,n_{p+q+1}) - a(U';n_{p+q+1}-k,n_{p+q+1})b(U'';n_0,n_{p+q+1}-k)],$$

where because of the assumptions on the support of a, b, the k sum is for indices satisfying $|k| \leq cn_0 \sim cn_{p+q+1}$ for some small constant $c > 0$ (see (2.1.11)). We may rewrite this using notations (2.2.6) and (2.2.8)

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}(\Delta a)(U';n_0,n_{p+q+1};k)b(U'';n_0+k,n_{p+q+1})\\ &\qquad+\sum_{k\in\mathbb{Z}}a(U';n_{p+q+1}-k,n_{p+q+1})\mathcal{S}((\partial_\xi-\partial_\eta^*)b)(U'';n_0,n_{p+q+1};k).\end{aligned}\tag{2.2.11}$$

We now prove estimates of type (2.1.12) for each k sum above. We start with the second one. If we evaluate the above symbol at $\Pi_{n'}U'$, $\Pi_{n''}U''$ instead of U', U'', we get from (2.2.7) and (2.1.12)

$$\begin{aligned}&\|\mathcal{S}((\partial_\xi-\partial_\eta^*)b)(\Pi_{n''}U'';n_0,n_{p+q+1};k)\|\\ &\leq C(1+|k|)(n_0+n_{p+q+1})^{d'-1}\frac{|n''|^{\nu+N+N_0}}{(|n_0-n_{p+q+1}+k|+|n''|)^N}\prod_{p+1}^{p+q}\|u_j\|_{L^2}.\end{aligned}$$

Moreover, if we make act derivatives on $\mathcal{S}((\partial_\xi-\partial_\eta^*)b)$, we have, because of (2.2.10) the same gains and losses as in (2.1.12). On the other hand, by (1.2.8), making act a $\partial_{n_{p+q+1}}$ derivative on $a(\Pi_{n'}U';n_{p+q+1}-k,n_{p+q+1})$ provides a gain of one negative power of n_{p+q+1}, and a loss of $|n'|^{N_0}$. Using (1.2.6), (1.2.7), we thus see that the action of $\partial_{n_0}^\alpha(\partial_{n_{p+q+1}}^*)^\beta(\partial_{n_0}-\partial_{n_{p+q+1}}^*)^\gamma$ on the general term of the second sum in (2.2.11) is bounded from above by $\prod_1^{p+q}\|u_j\|_{L^2}$ times

$$C(1+|k|)n_{p+q+1}^{d_1}\frac{|n'|^{\nu+N_1+\kappa_1}}{(|k|+|n'|)^{N_1}}\frac{(n_0+n_{p+q+1})^{d_2}|n''|^{\nu+N_2+\kappa_2}}{(|n_0-n_{p+q+1}+k|+|n''|)^{N_2}}\tag{2.2.12}$$

with $d_1+d_2 = d+d'-1-\gamma$, $\kappa_1+\kappa_2 = (\alpha+\beta+\gamma+1)N_0$, N_1, N_2 arbitrary. It is clear that the sum in k satisfying $|k| \ll n_0 \sim n_{p+q+1}$ of these quantities is bounded from above by

$$C(n_0+n_{p+q+1})^{d+d'-1-\gamma}\frac{(|n'|+|n''|)^{2\nu+3+N_0(\alpha+\beta+\gamma+1)+N}}{(|n_0-n_{p+q+1}|+|n'|+|n''|)^N}\tag{2.2.13}$$

which is the (2.1.12)-like estimate wanted (with ν replaced by $\nu' = 2\nu+3+N_0$). Let us study now the first sum in (2.2.11). It follows from (2.2.7), (2.2.8) and the fact that $|k| \ll n_0 \sim n_{p+q+1}$ that

$$\|(\Delta a)(\Pi_{n'}U';n_0,n_{p+q+1};k)\| \leq C(1+|n_0-n_{p+q+1}+k|)(n_0+n_{p+q+1})^{d-1}\frac{|n'|^{\nu+N+N_0}}{(|k|+|n'|)^N}\prod_1^p\|u_j\|_{L^2}.$$

Moreover, if we make act $\partial_{n_0}-\partial_{n_{p+q+1}}^*$ on Δa, we gain because of (2.2.9) a decay of type $(n_0+n_{p+q+1})^{-1}$, and loose $|n'|^{N_0}$. In the same way, ∂_{n_0} or $\partial_{n_{p+q+1}}^*$ loose $|n'|^{N_0}$. Similar properties hold true when derivatives act on $b(\Pi_{n''}U'';n_0+k,n_{p+q+1})$. Consequently, using Leibniz formulas

(1.2.6), we see that the action of $\partial_{n_0}^{\alpha}(\partial^*_{n_{p+q+1}})^{\beta}(\partial_{n_0}-\partial^*_{n_{p+q+1}})^{\gamma}$ on the general term of the first sum (2.2.11) gives a quantity bounded from above by an expression similar to (2.2.12), but where k has been replaced by $-k-n_0+n_{p+q+1}$. We obtain as above that the k-sum is then estimated by (2.2.13). This concludes the proof. □

Let us study now composition relatively to an inner argument.

Proposition 2.2.3 *Let $d'\in\mathbb{R},\nu\in\mathbb{R}_+,N_0\in\mathbb{N}^*$. There is $\nu'=2\nu+d'_++1$ such that for any $p\in\mathbb{N},q\in\mathbb{N}^*,d\in\mathbb{R}$, for any $a\in\Sigma^{d,\nu}_{q,N_0}$, $b\in\Sigma^{d',\nu}_{p,N_0}$ satisfying condition (i)$_\delta$ of definition 2.1.1 with a small enough $\delta>0$, there is $c\in\Sigma^{d,\nu'}_{p+q,N_0}$ such that for any $U=(U^{(1)},U^{(2)})\in\mathcal{E}^{p+q}$ with $U^{(1)}=(u_1,\dots,u_p)$, $U^{(2)}=(u_{p+1},U^{(3)})$, $U^{(3)}=(u_{p+2},\dots,u_{p+q})$, for any $u_{p+q+1}\in\mathcal{E}$, one has*

$$\mathrm{Op}[a(\mathrm{Op}(b(U^{(1)};\cdot))u_{p+1},U^{(3)};\cdot)]u_{p+q+1}=\mathrm{Op}(c(U^{(1)},U^{(2)};\cdot))u_{p+q+1}. \tag{2.2.14}$$

Proof: By definition 2.1.2, we may write the left hand side as

$$\sum_{n_0}\sum_{n_{p+q+1}}\sum_{k}\sum_{n_{p+1}}\mathcal{F}^*_{n_0}a\big[\mathcal{F}^*_kb(U^{(1)};k,n_{p+1})\mathcal{F}_{n_{p+1}}u_{p+1},U^{(3)};n_0,n_{p+q+1}\big]\mathcal{F}_{n_{p+q+1}}u_{p+q+1}$$

which is of form $\mathrm{Op}(c(U^{(1)},U^{(2)};\cdot))u_{p+q+1}$ if we define

$$c(U^{(1)},U^{(2)};n_0,n_{p+q+1})=\sum_{k}\sum_{n_{p+1}}a\big[\mathcal{F}^*_kb(U^{(1)};k,n_{p+1})\mathcal{F}_{n_{p+1}}u_{p+1},U^{(3)};n_0,n_{p+q+1}\big].$$

Let us check that if we denote by $n^{(1)}=(n_1,\dots,n_p)$, $n^{(2)}=(n_{p+1},n^{(3)})$, $n^{(3)}=(n_{p+2},\dots,n_{p+q})$, $c(\Pi_{n^{(1)}}U^{(1)},\Pi_{n^{(2)}}U^{(2)};n_0,n_{p+q+1})$ satisfies the conditions of definition 2.1.1. The support condition (i)$_{2\delta}$ holds true if (i)$_\delta$ is verified by a,b with small enough $\delta>0$. Moreover, it is enough to check (2.1.12) when $\alpha=\beta=\gamma=0$. Using the assumption on a,b, we get for $\|c(\Pi_{n^{(1)}}U^{(1)},\Pi_{n^{(2)}}U^{(2)};n_0,n_{p+q+1})\|$ an upper bound given by the product of $C\prod_{j=1}^{p+q}\|u_j\|_{L^2}$ and of

$$\sum_k(n_0+n_{p+q+1})^d\frac{(k+|n^{(3)}|)^{\nu+N_1}}{(|n_0-n_{p+q+1}|+|n^{(3)}|+k)^{N_1}}(n_{p+1}+k)^{d'}\frac{|n^{(1)}|^{\nu+N_2}}{(|k-n_{p+1}|+|n^{(1)}|)^{N_2}}$$

for any N_1,N_2. Moreover, by condition (i)$_\delta$ verified by a, b, the k-summation is made for $n_{p+1}\sim k\ll n_0\sim n_{p+q+1}$. We see that taking $N_2=0$, we get for the sum the upper bound

$$C(n_0+n_{p+q+1})^d\frac{|(n^{(1)},n^{(2)})|^{2\nu+d'_++N_1+1}}{(|n_0-n_{p+q+1}|+|(n^{(1)},n^{(2)})|)^{N_1}}$$

which gives the wanted conclusion with $\nu'=2\nu+d'_++1$. □

We shall study now composition of an operator associated to a paradifferential symbol with a remainder operator.

Proposition 2.2.4 *Let $p \in \mathbb{N}^*, q \in \mathbb{N}, d, d' \in \mathbb{R}, \nu \in \mathbb{R}_+, N_0 \in \mathbb{N}^*$. There are $\nu' = 2\nu + d'_+ + 1$ and $\nu'' = 2\nu + 1$ such that for any $a \in \Sigma^{d,\nu}_{q+1,N_0}$ satisfying condition (i)$_\delta$ of definition 2.1.1 with $\delta > 0$ small enough, for any $M \in \mathcal{R}^{d',\nu}_p$, there are a symbol $b \in \Sigma^{d,\nu'}_{p+q,N_0}$ and an operator $R \in \mathcal{R}^{d+d'_+,\nu''}_{p+q+1}$, such that for any $U = (U', u_{p+q+1}) \in \mathcal{E}^{p+q+1}$ with $U' = (U^{(1)}, U^{(2)})$, $U^{(1)} = (u_1, \dots, u_p)$, $U^{(2)} = (u_{p+1}, \dots, u_{p+q})$,*

$$\mathrm{Op}[a(M(U^{(1)}), U^{(2)}; \cdot)]u_{p+q+1} = \mathrm{Op}(b(U'; \cdot))u_{p+q+1} + R(U). \tag{2.2.15}$$

We shall use several times below an inequality established in the proof of theorem 2.1.4 of [10] (formulas (2.1.10) and (2.1.11) of that paper). We state this result as a separate lemma.

Lemma 2.2.5 *Let $\nu_1, \nu_2 \in \mathbb{R}_+$. There is, for any $N > 1 + \max(\nu_1, \nu_2)$, a constant $C_N > 0$ such that for any $n_0, \dots, n_{p+q+1} \in \mathbb{N}$,*

$$\sum_k \frac{\mu(n_0, \dots, n_p, k)^{\nu_1+N}}{S(n_0, \dots, n_p, k)^N} \frac{\mu(k, n_{p+1}, \dots, n_{p+q+1})^{\nu_2+N}}{S(k, n_{p+1}, \dots, n_{p+q+1})^N} \tag{2.2.16}$$

is bounded from above by

$$C_N \frac{\mu(n_0, \dots, n_{p+q+1})^{\nu'+N'}}{S(n_0, \dots, n_{p+q+1})^{N'}} \tag{2.2.17}$$

where $N' = N - 1 - \max(\nu_1, \nu_2), \nu' = \nu_1 + \nu_2 + 1$.

Proof of proposition 2.2.4: Let $\chi \in C_0^\infty(\mathbb{R}), \chi \equiv 1$ close to zero, $0 \leq \chi \leq 1$ with $\mathrm{Supp}\,\chi$ small enough. If for $n = (n_0, \dots, n_{p+q+1})$ we set $n^{(1)} = (n_1, \dots, n_p)$, $n^{(2)} = (n_{p+1}, \dots, n_{p+q})$, $n' = (n^{(1)}, n^{(2)})$, we define

$$b(U'; n_0, n_{p+q+1}) = \sum_{n^{(1)}} \chi\Big(\frac{|n^{(1)}|}{n_0 + n_{p+q+1}}\Big) a(M(\Pi_{n^{(1)}} U^{(1)}), U^{(2)}; n_0, n_{p+q+1}). \tag{2.2.18}$$

Remark that if $\mathrm{Supp}\,\chi$ is small enough, condition (i) of definition 2.1.1 will be satisfied by b. We use (2.1.12) for a to estimate $\|b(\Pi_{n'} U'; n_0, n_{p+q+1})\|$ by

$$C(n_0 + n_{p+q+1})^d \sum_k \frac{(k + |n^{(2)}|)^{\nu+N}}{(|n_0 - n_{p+q+1}| + k + |n^{(2)}|)^N} \|\Pi_k M(\Pi_{n^{(1)}} U^{(1)})\|_{L^2} \prod_{p+1}^{p+q} \|u_j\|_{L^2}$$

where the summation is made for $k + |n^{(2)}| \ll n_0 \sim n_{p+q+1}$, and where moreover $|n^{(1)}| \ll n_0 \sim n_{p+q+1}$. In other words, using notation (2.1.5), we may write the first factor in the k-sum as,

$$\frac{\mu(n_0, k, n^{(2)}, n_{p+q+1})^{\nu+N}}{S(n_0, k, n^{(2)}, n_{p+q+1})^N}.$$

We estimate the second factor using (2.1.15). We get for any N an upper bound given by the product of $\prod_1^{p+q} \|u_j\|_{L^2}$ and of

$$C(n_0 + n_{p+q+1})^d (1 + |n^{(1)}|)^\nu \sum_k \frac{\mu(n_0, k, n^{(2)}, n_{p+q+1})^{d'_+ + \nu + N}}{S(n_0, k, n^{(2)}, n_{p+q+1})^N} \frac{\mu(k, n^{(1)})^N}{S(k, n^{(1)})^N}.$$

By lemma 2.2.5, we obtain the bound

$$C(n_0+n_{p+q+1})^d(1+|n^{(1)}|)^\nu \frac{\mu(n_0,n^{(1)},n^{(2)},n_{p+q+1})^{d'_++\nu+N'+1}}{S(n_0,n^{(1)},n^{(2)},n_{p+q+1})^{N'}}.$$

Since we have by assumption $|n^{(2)}| \ll n_0 \sim n_{p+q+1}$, and on the support of the cut-off (2.2.18) $|n^{(1)}| \ll n_0$, we see that

$$\mu(n_0,n^{(1)},n^{(2)},n_{p+q+1}) = |n'|, S(n_0,n^{(1)},n^{(2)},n_{p+q+1}) = |n_0-n_{p+q+1}|+|n'|.$$

We thus get for b an estimate of form (2.1.12) since derivatives are controlled in the same way. The remainder in (2.2.15) will be given by

$$\begin{aligned}R(U) = \sum_{n^0}\sum_{n^{(1)}}\sum_{n^{(2)}}\sum_{n_{p+q+1}} &\chi_1\Big(\frac{|n^{(1)}|}{n_0+n_{p+q+1}}\Big)\\ &\times \mathcal{F}^*_{n_0}\big[a(\Pi_k M(\Pi_{n^{(1)}}U^{(1)}),\Pi_{n^{(2)}}U^{(2)};n_0,n_{p+q+1})\mathcal{F}_{n_{p+q+1}}u_{p+q+1}\big],\end{aligned} \tag{2.2.19}$$

where $\chi_1 = 1-\chi$. The L^2 norm of $\Pi_{n_0}R(\Pi_{n^{(1)}}U^{(1)},\Pi_{n^{(2)}}U^{(2)},\Pi_{n_{p+q+1}}u_{p+q+1})$ will be bounded from above using definitions 2.1.1 and 2.1.4 by $\prod_1^{p+q+1}\|u_j\|_{L^2}$ times

$$\begin{aligned}\sum_k \chi_1\Big(\frac{|n^{(1)}|}{n_0+n_{p+q+1}}\Big)&\frac{(k+|n^{(2)}|)^{\nu+N}}{(|n_0-n_{p+q+1}|+|n^{(2)}|+k)^N}(n_0+n_{p+q+1})^d\\ &\times k^{d'}\frac{(\max_2(n^{(1)}))^{\nu+\ell}}{(\max(n^{(1)}))^\ell}\frac{\mu(k,n^{(1)})^N}{S(k,n^{(1)})^N}\end{aligned} \tag{2.2.20}$$

and because of condition (i) of definition 2.1.1, we may restrict the summation to those k satisfying $k+|n^{(2)}| \ll n_0 \sim n_{p+q+1}$. Moreover, the cut-off χ_1 localizes for $|n^{(1)}| \geq cn_0$. Consequently (2.2.20) will be bounded from above by

$$Cn_0^{d+d'_+}\frac{(\max_2(n^{(1)}))^{\nu+\ell}}{\max(n_0,n^{(1)},n^{(2)},n_{p+q+1})^\ell}\sum_k \frac{\mu(n_0,k,n^{(2)},n_{p+q+1})^{\nu+N}}{S(n_0,k,n^{(2)},n_{p+q+1})^N}\frac{\mu(k,n^{(1)})^N}{S(k,n^{(1)})^N}.$$

Using again lemma 2.2.5, we get an upper bound

$$Cn_0^{d+d'_+}\frac{\max_2(n_1,\dots,n_{p+q+1})^{\nu''+\ell}}{\max(n_1,\dots,n_{p+q+1})^\ell}\frac{\mu(n_0,\dots,n_{p+q+1})^{N''}}{S(n_0,\dots,n_{p+q+1})^{N''}}$$

for new values $\nu''=2\nu+1, N''$ of ν, N. This is the wanted remainder estimate. $\square$

Let us study now the action of an operator on a remainder.

Proposition 2.2.6 *Let $p\in\mathbb{N}, q\in\mathbb{N}^*, d\in\mathbb{R}, d'\in\mathbb{R}, \nu\in\mathbb{R}_+, \nu'\in\mathbb{R}_+, N_0\in\mathbb{N}^*$. There is $\nu''=\nu+\nu'+1$ such that for any $a\in\Sigma^{d,\nu}_{p,N_0}$, any $M\in\mathcal{R}^{d',\nu'}_q$, the operator*

$$(u_1,\dots,u_{p+q}) \to \mathrm{Op}(a(u_1,\dots,u_p;\cdot))M(u_{p+1},\dots,u_{p+q}) \tag{2.2.21}$$

is in $\mathcal{R}^{d+d',\nu''}_{p+q}$.

Proof: We denote by $U^{(1)} = (u_1, \dots, u_p)$, $U^{(2)} = (u_{p+1}, \dots, u_{p+q})$, $n^{(1)} = (n_1, \dots, n_p)$, $n^{(2)} = (n_{p+1}, \dots, n_{p+q})$. The value of operator (2.2.21) at $(\Pi_{n^{(1)}} U^{(1)}, \Pi_{n^{(2)}} U^{(2)})$ is

$$\sum_{n_0} \sum_k \mathcal{F}^*_{n_0}[a(\Pi_{n^{(1)}} U^{(1)}; n_0, k)\mathcal{F}_k M(\Pi_{n^{(2)}} U^{(2)})].$$

We make act Π_{n_0} on this expression, and compute the L^2-norm. Using definitions 2.1.1 and 2.1.4, we get an estimate in terms of the product of $\prod_1^{p+q} \|u_j\|_{L^2}$ by

$$C \sum_k (n_0 + k)^d \frac{|n^{(1)}|^{\nu+N}}{(|n_0 - k| + |n^{(1)}|)^N} k^{d'} \frac{(\max_2 n^{(2)})^{\nu'+\ell}}{(\max n^{(2)})^\ell} \frac{\mu(k, n^{(2)})^N}{S(k, n^{(2)})^N} \tag{2.2.22}$$

and we have on the support of the summation $k \sim n_0 \gg |n^{(1)}|$.

• If moreover $k \sim n_0 \gg |n^{(2)}|$, we get for (2.2.22) an estimate

$$C \sum_k n_0^{d+d'} |n^{(1)}|^\nu \frac{(\max_2 n^{(2)})^{\nu'+N}}{n_0^N}.$$

Since we sum for $|k - n_0| \le cn_0$ by condition (i) of definition 2.1.1, this gives the upper bound

$$Cn_0^{d+d'+1+\nu-N} (\max_2 n^{(2)})^{\nu'+N} \le Cn_0^{d+d'} \frac{(\max_2(n^{(1)}, n^{(2)}))^{\nu+\nu'+1+\ell'}}{(\max(n^{(1)}, n^{(2)}))^{\ell'}} \frac{\mu(n_0, \dots, n_{p+q})^{N'}}{S(n_0, \dots, n_{p+q})^{N'}}$$

if we take $N = \ell' + N' + \nu + 1$. This is a remainder type estimate.

• If $|n^{(2)}| \ge cn_0$ for some $c > 0$, we bound (2.2.22) from above by

$$Cn_0^{d+d'} \frac{(\max_2(n^{(1)}, n^{(2)}))^{\nu'+\ell}}{(\max(n^{(1)}, n^{(2)}))^\ell} \sum_k \frac{\mu(n_0, n^{(1)}, k)^{\nu+N}}{S(n_0, n^{(1)}, k)^N} \frac{\mu(k, n^{(2)})^N}{S(k, n^{(2)})^N}.$$

Using again lemma 2.2.5 to estimate the k-sum, we obtain finally in this case

$$Cn_0^{d+d'} \frac{(\max_2(n^{(1)}, n^{(2)}))^{\nu'+\ell}}{(\max(n^{(1)}, n^{(2)}))^\ell} \frac{\mu(n_0, n^{(1)}, n^{(2)})^{\nu+N'+1}}{S(n_0, n^{(1)}, n^{(2)})^{N'}}$$

for a new N'. This implies the wanted remainder estimate. □

Proposition 2.2.7 *Let $d, d' \in \mathbb{R}, \nu, \nu' \in \mathbb{R}_+$.*

(i) Let $p \in \mathbb{N}, q \in \mathbb{N}^*, N_0 \in \mathbb{N}^*$. There is $\nu'' = d_+ + \nu + \nu' + 1$ such that for any $a \in \Sigma^{d,\nu}_{p,N_0}$, any $M \in \mathcal{R}^{d',\nu'}_q$ the operator

$$R(u_1, \dots, u_{p+q}) = M(\mathrm{Op}(a(u_1, \dots, u_p; \cdot))u_{p+1}, u_{p+2}, \dots, u_{p+q}) \tag{2.2.23}$$

belongs to $\mathcal{R}^{d',\nu''}_{p+q}$.

(ii) Let $p \in \mathbb{N}^*, q \in \mathbb{N}^*$. There is $\nu'' = \nu + \nu' + 1 + d'_+$ such that for any $M_1 \in \mathcal{R}^{d,\nu}_q, M_2 \in \mathcal{R}^{d',\nu'}_p$ the operator

$$(u_1, \dots, u_{p+q-1}) \to M_1(M_2(u_1, \dots, u_p), u_{p+1}, \dots, u_{p+q-1}) \tag{2.2.24}$$

belongs to $\mathcal{R}^{d,\nu''}_{p+q-1}$.

Proof: (i) Denoting again $U^{(1)} = (u_1, \dots, u_p)$, $U^{(2)} = (u_{p+1}, U^{(3)})$, $U^{(3)} = (u_{p+2}, \dots, u_{p+q})$, and using similar notations $n^{(1)}, n^{(2)}, n^{(3)}$ for the indices, we have to estimate the quantity

$$\sum_k \Pi_{n_0} M(\mathcal{F}_k^* a(\Pi_{n^{(1)}} U^{(1)}; k, n_{p+1}) \mathcal{F}_{n_{p+1}} u_{p+1}, \Pi_{n^{(3)}} U^{(3)}). \tag{2.2.25}$$

The L^2-norm of the general term of (2.2.25) is bounded from above by $\prod_1^{p+q} \|u_j\|_{L^2}$ times

$$Cn_0^{d'} \frac{\max_2(k, n^{(3)})^{\nu'+\ell}}{\max(k, n^{(3)})^\ell} \frac{\mu(n_0, k, n^{(3)})^N}{S(n_0, k, n^{(3)})^N} (k + n_{p+1})^d \frac{|n^{(1)}|^{\nu+N}}{(|k - n_{p+1}| + |n^{(1)}|)^N}.$$

Moreover the summation is restricted to $|n^{(1)}| \ll k \sim n_{p+1}$, which allows one to bound this quantity by

$$Cn_0^{d'} n_{p+1}^d \frac{\max_2(n^{(1)}, n^{(2)})^{\nu'+\ell}}{\max(n^{(1)}, n^{(2)})^\ell} \frac{\mu(n_0, k, n^{(3)})^N}{S(n_0, k, n^{(3)})^N} \frac{\mu(k, n^{(1)}, n_{p+1})^{\nu+N}}{S(k, n^{(1)}, n_{p+1})^N}.$$

Using again lemma 2.2.5 to estimate the k-sum, we get an expression of type

$$Cn_0^{d'} \frac{\max_2(n^{(1)}, n^{(2)})^{\nu''+\ell}}{\max(n^{(1)}, n^{(2)})^\ell} \frac{\mu(n_0, n^{(1)}, n^{(2)})^N}{S(n_0, n^{(1)}, n^{(2)})^N}$$

for $\nu'' = d_+ + \nu + \nu' + 1$, and new values of N, ℓ.

(ii) We need to estimate the L^2-norm of

$$\sum_k \Pi_{n_0} M_1[\Pi_k M_2(\Pi_{n^{(1)}} U^{(1)}), \Pi_{n^{(2)}} U^{(2)}] \tag{2.2.26}$$

if we denote here $U^{(1)} = (u_1, \dots, u_p)$, $U^{(2)} = (u_{p+1}, \dots, u_{p+q-1})$ and use similar notations for $n^{(1)}, n^{(2)}$. The L^2-norm of the general term of (2.2.26) is bounded from above by

$$Cn_0^d \frac{\max_2(k, n^{(2)})^{\nu+\ell_2}}{\max(k, n^{(2)})^{\ell_2}} \frac{\mu(n_0, k, n^{(2)})^{N_2}}{S(n_0, k, n^{(2)})^{N_2}} k^{d'} \frac{\max_2(n^{(1)})^{\nu'+\ell_1}}{\max(n^{(1)})^{\ell_1}} \frac{\mu(k, n^{(1)})^{N_1}}{S(k, n^{(1)})^{N_1}}.$$

Assume for instance $n_1 \leq \cdots \leq n_p$, $n_{p+q-1} \leq \cdots \leq n_{p+1}$. The above expression may be written

$$\begin{aligned} &Cn_0^d k^{d'} \frac{n_{p-1}^{\nu'+\ell_1}}{n_p^{\ell_1}} \frac{\max_2(k, n_{p+2}, n_{p+1})^{\nu+\ell_2}}{\max(k, n_{p+1})^{\ell_2}} \\ &\qquad\times \frac{\mu(k, n_{p-2}, n_{p-1}, n_p)^{N_1}}{S(k, n_{p-2}, n_{p-1}, n_p)^{N_1}} \frac{\mu(n_0, k, n_{p+3}, n_{p+2}, n_{p+1})^{N_2}}{S(n_0, k, n_{p+3}, n_{p+2}, n_{p+1})^{N_2}}. \end{aligned} \tag{2.2.27}$$

Remark first that, changing eventually the definition of ℓ_2, we can control the $k^{d'}$ term by $\max_2(k, n_{p+2}, n_{p+1})^{d'_+}$. In the following we thus remove the $k^{d'}$ term and replace ν by $\nu + d'_+$.

• If $n_p \geq \frac{1}{A} n_{p+1}$ for a large enough constant $A > 0$, we take $\ell_1 = \ell, \ell_2 = 0$ and we get an upper bound of type

$$Cn_0^d \frac{\max_2(n^{(1)}, n^{(2)})^{\nu+\nu'+d'_++\ell}}{\max(n^{(1)}, n^{(2)})^\ell} \frac{\mu(k, n^{(1)})^{N_1}}{S(k, n^{(1)})^{N_1}} \frac{\mu(n_0, k, n^{(2)})^{N_2}}{S(n_0, k, n^{(2)})^{N_2}}. \tag{2.2.28}$$

• If $k \leq An_p < n_{p+1}$, we see that in (2.2.27),

$$\max_2(k, n_{p+2}, n_{p+1}) \leq A(n_p + n_{p+2}) \leq C \max_2(n^{(1)}, n^{(2)}).$$

We take $\ell_1 = 0, \ell_2 = \ell$ and get again an estimate by (2.2.28).

• If $n_p < \frac{1}{A} n_{p+1}$ and $n_p < \frac{1}{A} k$, the last but one factor in (2.2.27) may be written $\frac{n_{p-1}^{N_1}}{k^{N_1}}$. Moreover $\max_2(k, n_{p+2}, n_{p+1}) \leq k$ if we assume $k \geq n_{p+2}$. Taking in (2.2.27) $\ell_1 = \ell_2 = 0$, $N_1' < N_1 - \nu - d_+'$, when $n_{p+1} \leq k$, and $\ell_1 = 0$, $\ell_2 = N_1' < N_1 - \nu - d_+'$ when $n_{p+1} > k$ we get the upper bound

$$Cn_0^d \frac{n_{p-1}^{\nu' + N_1' + \nu + d_+'}}{n_{p+1}^{N_1'}} \frac{\mu(k, n^{(1)})^{N_1 - N_1' - \nu - d_+'}}{S(k, n^{(1)})^{N_1 - N_1' - \nu - d_+'}} \frac{\mu(n_0, k, n^{(2)})^{N_2}}{S(n_0, k, n^{(2)})^{N_2}}$$

which again gives an estimate of type (2.2.28) (changing the definition of the exponents).

If $k < n_{p+2}$, we take in (2.2.27) $\ell_1 = 0$ and get an estimate by

$$Cn_0^d \frac{n_{p+2}^{\nu + d_+' + \ell_2}}{n_{p+1}^{\ell_2}} n_{p-1}^{\nu'} \frac{\mu(k, n^{(1)})^{N_1}}{S(k, n^{(1)})^{N_1}} \frac{\mu(n_0, k, n^{(2)})^{N_2}}{S(n_0, k, n^{(2)})^{N_2}}.$$

We get again an estimate of type (2.2.28). To finish the proof, we just have to sum (2.2.28) using again lemma 2.2.5 to get the wanted upper bound

$$Cn_0^d \frac{\max_2(n^{(1)}, n^{(2)})^{\nu'' + \ell}}{\max(n^{(1)}, n^{(2)})^{\ell}} \frac{\mu(n_0, n^{(1)}, n^{(2)})^N}{S(n_0, n^{(1)}, n^{(2)})^N}$$

with $\nu'' = \nu + \nu' + d_+' + 1$. □

3 Special pseudo-differential operators

3.1 An introductory example

In addition to the paradifferential symbols introduced in section 2, we shall need classes of pseudo-differential operators. These classes will be more peculiar than the corresponding para-differential ones. Let us explain this, and justify their definition through an example. Assume that we are given an orthogonal decomposition $L^2 = \bigoplus E_n$, and assume that E_n is one dimensional, spanned by a normalized eigenfunction φ_n. Let $(X, n) \to b(X, n)$ be a linear real valued function of $X \in \mathbb{R}$, which is a symbol of order 0 relatively to n ($\partial_n^\alpha b(X, n) = O(n^{-\alpha})$, $n \to +\infty$). If $u_1 \in \mathcal{E}$, we can define the action of the pseudo-differential operator with symbol $b(u_1, n)$ on a function u_2 by the formula

(3.1.1) $$\sum_{n_2} b(u_1, n_2) \langle u_2, \varphi_{n_2} \rangle \varphi_{n_2}.$$

We denote, for future generalization, by $B(X, n)$ the map from E_n to E_n given for any fixed $X \in \mathbb{R}$ by

$$B(X, n) : \varphi_n \to b(X, n)\varphi_n \tag{3.1.2}$$

so that (3.1.1) may be written, if we remember that the orthogonal projection on E_n is given by $\Pi_n u = \langle u, \varphi_n \rangle \varphi_n$,

$$\sum_{n_2} B(u_1, n_2)\Pi_{n_2} u_2. \tag{3.1.3}$$

Remark also that (3.1.1) may be rewritten

$$\sum_{n_0} \sum_{n_2} \varphi_{n_0}(x)\langle b(u_1, n_2)\varphi_{n_2}, \varphi_{n_0}\rangle\langle u_2, \varphi_{n_2}\rangle \tag{3.1.4}$$

i.e. with notations (2.1.6), (2.1.7)

$$\sum_{n_0} \sum_{n_2} \mathcal{F}^*_{n_0} c(u_1; n_0, n_2)\mathcal{F}_{n_2} u_2 \tag{3.1.5}$$

with

$$c(u_1; n_0, n_2) = \langle b(u_1, n_2)\varphi_{n_2}, \varphi_{n_0}\rangle. \tag{3.1.6}$$

In other words, the operator (3.1.1) may be written under form (2.1.13) with a symbol c which may be proved to satisfy estimates (2.1.12).

Our aim in this third section is to introduce a general class of operators of form (3.1.1). We shall see that they may be expressed in terms of quantities like (3.1.5) i.e. from (a sum of) paradifferential operators associated to symbols of the classes $\Sigma^{d,\nu}_{p,N_0}$ studied in section 2, up to remainder operators. The interest of operators defined through formula (3.1.1) instead of (3.1.5), is that they obey more explicit calculus rules, in particular for the symbol of the composition of two operators. On the other hand, we do not escape the necessity of introducing more general operators, of form (3.1.5), since to prove our main theorem, we shall have to define from operators of type (3.1.1) more general ones, given by symbols of type (3.1.6).

3.2 Definition and calculus of special symbols

Remind that we denoted at the beginning of subsection 2.1 by G a finite dimensional real vector space. Let $(g_i)_i$ be a basis of G. We fix a nice basis $(\varphi^j_n)_{n,j}$ of $L^2(\mathbb{S}^1, \mathbb{R})$, where $(\varphi^j_n)_j$ is a basis of the subspace E'_n generated by the eigenfunctions associated to the eigenvalues $\omega_-(n) \leq \omega_+(n)$ of $\sqrt{-\Delta + V}$. For $\ell = (j, i)$ we set $\varphi^\ell_n = \varphi^j_n \otimes g_i$. Then $(\varphi^\ell_n)_\ell$ is a basis of $E_n = E'_n \otimes G$ and $(\varphi^\ell_n)_{n,\ell}$ is a nice basis of $L^2(\mathbb{S}^1, G) \simeq L^2(\mathbb{S}^1, \mathbb{R}) \otimes G$, and we have $L^2(\mathbb{S}^1, G) = \bigoplus_{n \geq \tau} E_n$. Of course $(\varphi^\ell_n)_\ell$ provides also a basis of $E_n \otimes \mathbb{C}$ and $(\varphi^\ell_n)_{\ell,n}$ is a nice basis of $L^2(\mathbb{S}^1, G \otimes \mathbb{C})$ considered as a $\mathbb{C}$-vector space.

Definition 3.2.1 *Let $d \in \mathbb{R}$, $p \in \mathbb{N}$. We denote by S_p^d the space of maps*

$$(3.2.1)\qquad \begin{aligned} &(u_1, \dots, u_p, n_{p+1}) \to b(u_1, \dots, u_p, n_{p+1}) \\ &\mathcal{E} \times \cdots \times \mathcal{E} \times \mathbb{N}_\tau \longrightarrow \mathcal{L}(\mathcal{E}, L^2(\mathbb{S}^1, G \otimes \mathbb{K})) \end{aligned}$$

such that one can find

- *A map*

$$B : G \times \cdots G \times \mathbb{N}_\tau \to \mathcal{L}(\mathcal{E} \otimes \mathbb{K}, \mathcal{E} \otimes \mathbb{K}), (X_1, \dots, X_p, n) \to B(X_1, \dots, X_p, n)$$

which is for any fixed value of n, p-linear in $(X_1, \dots, X_p)$, such that for any $X_1, \dots, X_p \in G$, any $n \in \mathbb{N}_\tau$, $B(X_1, \dots, X_p, n)$ is an element of $\mathcal{L}(E_n \otimes \mathbb{K}, E_n \otimes \mathbb{K})$ (extended by zero on $(E_n \otimes \mathbb{K})^\perp$), whose matrix elements in the nice basis $(\varphi_n^\ell)_\ell$ of $E_n \otimes \mathbb{K}$ satisfy for any $\alpha \in \mathbb{N}$

$$(3.2.2)\qquad |\partial_n^\alpha B_{\ell\ell'}(X_1, \dots, X_p, n)| \le C_\alpha n^{d-\alpha} \prod_1^p |X_j|_G,$$

- *A family of pseudo-differential operators of order 0 on $\mathbb{S}^1$, $T_1, \dots, T_p$, such that one may write for any $u_1, \dots, u_p \in \mathcal{E}$, $n_{p+1} \in \mathbb{N}_\tau$*

$$(3.2.3)\qquad b(u_1, \dots, u_p, n_{p+1}) = B(T_1 u_1, \dots, T_p u_p, n_{p+1}).$$

We shall quantize the above operators in the following way:

Definition 3.2.2 *Let $b \in S_p^d$. We define an operator $\widetilde{\mathrm{Op}}(b)$ acting on $\mathcal{E}^{p+1}$ by*

$$(3.2.4)\qquad \widetilde{\mathrm{Op}}(b)(u_1, \dots, u_p, \cdot)u_{p+1} = \sum_{n_{p+1}} b(u_1, \dots, u_p, n_{p+1}) \Pi_{n_{p+1}} u_{p+1}.$$

We want now to define from an element of S_p^d and from a cut-off function a symbol in the class $\Sigma_{p,1}^{d,\nu}$.

Proposition 3.2.3 *Let $\chi \in C_0^\infty(\mathbb{R})$, χ even with small enough support, $p \in \mathbb{N}^*$. There is $\nu \in \mathbb{R}_+$ such that for any $d \in \mathbb{R}$, if we define for $b \in S_p^d$, $u_1, \dots, u_p \in \mathcal{E}$, $n_0, n_{p+1} \in \mathbb{N}_\tau$*
(3.2.5)

$$b_\chi(u_1, \dots, u_p; n_0, n_{p+1}) = \sum_{n_1} \cdots \sum_{n_p} \chi\Big(\frac{|n'|}{n_0 + n_{p+1}}\Big) \chi\Big(\frac{n_0 - n_{p+1}}{n_0 + n_{p+1}}\Big) \mathcal{F}_{n_0} \circ b(\Pi_{n'} U', n_{p+1}) \circ \mathcal{F}_{n_{p+1}}^*$$

where $U' = (u_1, \dots, u_p)$, $n' = (n_1, \dots, n_p)$, then $b_\chi \in \Sigma_{p,1}^{d,\nu}$. When $p = 0$, we shall set $b_\chi(n_0, n_1) = \mathcal{F}_{n_0} \circ b(n_1) \circ \mathcal{F}_{n_1}^*$, which is supported for $n_0 = n_1$.

Remark We assume in the statement that χ is even since this implies when, in (3.2.3), $B(X)$ is a self-adjoint linear map independent of n_{p+1}, that the symbol b_χ defined by (3.2.5) is self-adjoint i.e. satisfies with notations (2.2.1) that $b_\chi^\bullet(U'; n_0, n_{p+1}) = b_\chi(U'; n_0, n_{p+1})$.

Proof of proposition 3.2.3: Remark first that condition $(\mathrm{i})_\delta$ of definition 2.1.1 is satisfied if $\operatorname{Supp}\chi$ is small enough. Remind that we set $K(n) = \dim E_n$. Since $\mathcal{F}_n$ sends the basis $(\varphi_n^\ell)_\ell$ of $E_n \otimes \mathbb{K}$ onto the canonical basis of $\mathbb{K}^{K(n)}$, the matrix of $b_\chi(\Pi_{n'}U'; n_0, n_{p+1})$ in the canonical basis of $\mathbb{K}^{K(n_{p+1})}$ and $\mathbb{K}^{K(n_0)}$ is

$$\chi\Big(\frac{|n'|}{n_0+n_{p+1}}\Big)\chi\Big(\frac{n_0-n_{p+1}}{n_0+n_{p+1}}\Big)\Big(\langle b(\Pi_{n'}U', n_{p+1})\varphi_{n_{p+1}}^{\ell_{p+1}}, \varphi_{n_0}^{\ell_0}\rangle\Big)_{\ell_0,\ell_{p+1}}. \tag{3.2.6}$$

Remind also that for n_0, n_{p+1} large enough, the size of this matrix is independent of n_0, n_{p+1}. Using (2.1.18) to estimate derivatives of the cut-offs, and Leibniz formulas (1.2.6), (1.2.7), we see that we just have to get estimates of type (2.1.12) for the matrix in (3.2.6). Decompose $X_j \in G$ on the basis $(g_i)_i$ of G as $X_j = \sum_i X_j^i g_i$. Then the entries of the matrix of the map $B(X_1,\dots,X_p,n_{p+1})$ in the nice basis $(\varphi_n^\ell)_\ell$ of $E_n \otimes \mathbb{K}$ may be decomposed as

$$B_{\ell'_{p+1}\ell_{p+1}}(X_1,\dots,X_p,n_{p+1}) = \sum_I B^I_{\ell'_{p+1}\ell_{p+1}}(n_{p+1})X^I$$

where we denote by I a p-tuple $I = (i_1,\dots,i_p)$, by $X^I = \prod_{j=1}^p X_j^{i_j}$, and by $B^I_{\ell'_{p+1}\ell_{p+1}}(n_{p+1})$ the quantity $B_{\ell'_{p+1}\ell_{p+1}}(g_{i_1},\dots,g_{i_p},n_{p+1})$. By (3.2.3)

$$\langle b(\Pi_{n'}U', n_{p+1})\varphi_{n_{p+1}}^{\ell_{p+1}}, \varphi_{n_0}^{\ell_0}\rangle = \sum_{\ell'_{p+1}}\sum_I \langle B^I_{\ell'_{p+1}\ell_{p+1}}(n_{p+1})(T\Pi_{n'}U')^I\varphi_{n_{p+1}}^{\ell'_{p+1}}, \varphi_{n_0}^{\ell_0}\rangle \tag{3.2.7}$$

where $T\Pi_{n'}U' = (T_1\Pi_{n_1}u_1,\dots,T_p\Pi_{n_p}u_p)$. Since $\ell'_{p+1} \in \{1,\dots,K(n_{p+1})\}$ and $K(n)$ is independent of $n \to +\infty$, and since I describes also a finite set, we actually just need to estimate each term of the above sum, namely

$$B^I_{\ell'_{p+1}\ell_{p+1}}(n_{p+1})\langle (T\Pi_{n'}U')^I\varphi_{n_{p+1}}^{\ell'_{p+1}}, \varphi_{n_0}^{\ell_0}\rangle. \tag{3.2.8}$$

We apply inequality (2.1.2) with $T_1 = T_2 = \mathrm{Id}$ to the bracket. We get the following estimate

$$\begin{aligned}&|\partial_{n_0}^\alpha(\partial^*_{n_{p+1}})^\beta(\partial_{n_0}-\partial^*_{n_{p+1}})^\gamma\langle\varphi_{n_0}^{\ell_0}, (T\Pi_{n'}U')^I\varphi_{n_{p+1}}^{\ell'_{p+1}}\rangle|\\ &\qquad\le C\langle n_0-n_{p+1}\rangle^{-N}(n_0+n_{p+1})^{-\gamma}\sup_{0\le k\le\alpha+\beta+\gamma+N+\nu}\|\partial^k[(T\Pi_{n'}U')^I]\|_{L^\infty}\end{aligned} \tag{3.2.9}$$

for any $\alpha,\beta,\gamma,N \in \mathbb{N}$. By Sobolev injection, and the L^2-boundedness of pseudo-differential operators of order 0, we get for the last term in the above formula the upper bound

$$C(1+|n'|)^{\alpha+\beta+\gamma+N+\nu}\prod_1^p\|u_j\|_{L^2}$$

for a new value of ν. If we combine (3.2.9) with (3.2.8) and (3.2.2), and use Leibniz formulas (1.2.6), (1.2.7), we see that (3.2.7) satisfies estimate (2.1.12) of definition of symbols (with $N_0 = 1$). This concludes the proof. □

We shall need estimates of type (2.1.12) for some functions of type (3.2.5), but depending on extra parameters. We state theses estimates as a corollary of the proof of proposition 3.2.3.

Corollary 3.2.4 *(i) Let $b \in S^d_p$. One has the following estimate for any indices $n_0, n' = (n_1, \dots, n_p), n_{p+1}, k$:*

$$\|\Pi_{n_0} b(\Pi_{n'}U', k)\Pi_{n_{p+1}}\|_{\mathcal{L}(L^2,L^2)} \leq C \frac{|n'|^{\nu+N}}{(|n_0 - n_{p+1}| + |n'|)^N} k^d \tag{3.2.10}$$

for some $\nu \in \mathbb{R}_+$, independent of d.

(ii) Let $B : G \times \cdots \times G \times \mathbb{N}_\tau \times \mathbb{N}_\tau \to \mathcal{L}(\mathcal{E} \otimes \mathbb{K}, \mathcal{E} \otimes \mathbb{K})$ be a function

$$(X_1, \dots, X_p, n_{p+1}, k) \to B(X_1, \dots, X_p; n_{p+1}, k),$$

p-linear in $(X_1, \dots, X_p)$, and such that $B(X_1, \dots, X_p; n_{p+1}, k)$ is an element of $\mathcal{L}(E_k \otimes \mathbb{K}, E_k \otimes \mathbb{K})$, whose matrix elements in the nice basis of $E_k \otimes \mathbb{K}$ satisfy instead of (3.2.2)

$$|\partial^{\alpha_1}_{n_{p+1}} \partial^{\alpha_2}_k B_{\ell\ell'}(X_1, \dots, X_p; n_{p+1}, k)| \leq C(n_{p+1} + k)^{d-\alpha_1-\alpha_2} \prod_1^p |X_j|_G. \tag{3.2.11}$$

Define as in (3.2.5),

$$b_\chi(u_1, \dots, u_p, n_{p+1}; n_0, k) = \sum_{n_1} \cdots \sum_{n_p} \chi\Big(\frac{|n'|}{n_0 + k}\Big)\chi\Big(\frac{n_0 - k}{n_0 + k}\Big)\mathcal{F}_{n_0} \circ b(\Pi_{n'}U'; n_{p+1}, k) \circ \mathcal{F}^*_k.$$

Then b_χ satisfies

$$\begin{aligned} &\|\partial^\alpha_{n_0}(\partial^*_k)^{\beta_1}(\partial^*_{n_{p+1}})^{\beta_2}(\partial_{n_0} - \partial^*_k - \partial^*_{n_{p+1}})^\gamma b_\chi(\Pi_{n'}U', n_{p+1}; n_0, k)\| \\ &\qquad \leq C(n_0 + k)^{d-\gamma-\beta_2} \frac{|n'|^{\nu+N+\alpha+\beta_1+\beta_2+\gamma}}{(|n_0 - k| + |n'|)^N} \prod_1^p \|u_j\|_{L^2} \end{aligned} \tag{3.2.12}$$

for some $\nu \in \mathbb{R}_+$, independent of d.

Proof: (i) The left hand side of (3.2.10) equals $\|\mathcal{F}_{n_0} b(\Pi_{n'}U', k)\mathcal{F}^*_{n_{p+1}}\|$ by (2.1.8) and (2.1.6), (2.1.7). Consequently (3.2.10) is nothing but (2.1.12) in the case $\alpha = \beta = \gamma = 0$, when the symbol b depends on an extra parameter k, instead of being a function of n_{p+1} as in (3.2.5). Estimate (3.2.10) follows from (3.2.7) to (3.2.9) in the proof of proposition 3.2.3, in which $B^I_{\ell'_{p+1}\ell_{p+1}}$ is evaluated at k instead of n_{p+1}.

(ii) One has just to replace in the proof of proposition 3.2.3 the reference to (3.2.2) by the reference to (3.2.11), k playing now the role of n_{p+1}. Remark that since in (3.2.12) $\partial_{n_{p+1}}$-derivatives act only on the $B^I_{\ell'_{p+1}\ell_{p+1}}$ term in (3.2.8), they gain one negative power of $k \sim n_0 + k$. □

Our next task will be to express a quantity of form $\widetilde{\mathrm{Op}}(b(u_1, \dots, u_p, \cdot))u_{p+1}$ in terms of the action of paradifferential operators on $u_1, u_2, \dots, u_{p+1}$ and of a remainder operator. This is, in our framework, analogous to Bony's paradecomposition of a product [4].

Proposition 3.2.5 *Let $p \in \mathbb{N}^*$. Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero, with small enough support. There is $\nu \in \mathbb{R}_+$ and for any $d \in \mathbb{R}$, any symbol $b \in S_p^d$, a family of symbols $b_j \in \Sigma_{p,1}^{0,\nu+d_+}$ $j = 1, \ldots, p$, and a remainder operator $M \in \mathcal{R}_{p+1}^{0,\nu+d_+}$ such that for any $u_1, \ldots, u_{p+1} \in \mathcal{E}$*

$$
\begin{aligned}
\widetilde{\mathrm{Op}}(b(u_1, \ldots, u_p, \cdot))u_{p+1} = \; & \mathrm{Op}(b_\chi(u_1, \ldots, u_p; \cdot))u_{p+1} \\
& + \sum_{j=1}^{p} \mathrm{Op}(b_j(u_1, \ldots, \hat{u}_j, \ldots, u_{p+1}; \cdot))u_j \\
& + M(u_1, \ldots, u_{p+1}).
\end{aligned}
\tag{3.2.13}
$$

When $p = 0$, we have $\widetilde{\mathrm{Op}}(b(\cdot))u_1 = \mathrm{Op}(b_\chi(u; \cdot))u_1$.

Proof: We first define the symbols b_j, and check that they belong to $\Sigma_{p,1}^{0,\nu+d_+}$. Define for $j = 1, \ldots, p$

$$
\chi_j(n_0, \ldots, n_{p+1}) = \chi\Big(\frac{|(n_1, \ldots, \widehat{n_j}, \ldots, n_{p+1})|}{n_0 + n_j}\Big)\chi\Big(\frac{n_0 - n_j}{n_0 + n_j}\Big)
\tag{3.2.14}
$$

so that on $\operatorname{Supp}\chi_j$ we have

$$
n_k \leq c(n_0 + n_j),\ k \in \{1, \ldots, p+1\} - \{j\},\ |n_0 - n_j| \leq c(n_0 + n_j)
\tag{3.2.15}
$$

for a small constant $c > 0$. Moreover, $\chi_j \equiv 1$ on a domain of type (3.2.15) when c is replaced by some smaller constant. We define a linear map $b_j(u_1, \ldots, \widehat{u_j}, \ldots, u_{p+1}; n_0, n_j)$ from $\mathbb{K}^{K(n_j)}$ to $\mathbb{K}^{K(n_0)}$ as

$$
V \to \sum_{n_k; k \in \{1, \ldots, p+1\} - \{j\}} \chi_j(n_0, \ldots, n_{p+1})\mathcal{F}_{n_0}[b(\Pi_{n_1}u_1, \ldots, \mathcal{F}_{n_j}^* V, \ldots, \Pi_{n_p}u_p, n_{p+1})\Pi_{n_{p+1}}u_{p+1}].
\tag{3.2.16}
$$

By (3.2.15), condition (i)$_\delta$ of definition 2.1.1 will be satisfied if $c > 0$ is small enough. We must check the estimates of condition (ii). To simplify notations, take from now on $j = 1$, and set $n' = (n'', n_{p+1})$, $n'' = (n_2, \ldots, n_p)$, $U' = (U'', u_{p+1})$, $U'' = (u_2, \ldots, u_p)$, $\Pi_{n'}U' = (\Pi_{n_2}u_2, \ldots, \Pi_{n_{p+1}}u_{p+1})$. Then for $V \in \mathbb{K}^{K(n_1)}$, $b_1(\Pi_{n'}U'; n_0, n_1)\cdot V$ is the product of the function $\chi_1(n_0, \ldots, n_{p+1})$ by the vector of $\mathbb{K}^{K(n_0)}$ with components

$$
\langle b(\mathcal{F}_{n_1}^* V, \Pi_{n''}U'', n_{p+1})\Pi_{n_{p+1}}u_{p+1}, \varphi_{n_0}^{\ell_0}\rangle_{\ell_0}.
\tag{3.2.17}
$$

We use expression (3.2.3) for b in terms of B. Let $(V_{\ell_1})_{\ell_1}$ be the coordinates of $\mathcal{F}_{n_1}^* V$ on $(\varphi_{n_1}^{\ell_1})_{\ell_1}$ i.e. using Einstein's conventions $\mathcal{F}_{n_1}^* V = V_{\ell_1}\varphi_{n_1}^{\ell_1}$. We may rewrite (3.2.17)

$$
\langle V_{\ell_1} B(T_1\varphi_{n_1}^{\ell_1}, T''\Pi_{n''}U'', n_{p+1})\Pi_{n_{p+1}}u_{p+1}, \varphi_{n_0}^{\ell_0}\rangle_{\ell_0}
$$

where $T''\Pi_{n''}U'' = (T_2\Pi_{n_2}u_2, \ldots, T_p\Pi_{n_p}u_p)$. In other words, the (ℓ_0, ℓ_1) entry of the matrix of $b_1(\Pi_{n'}U'; n_0, n_1)$ in the canonical basis is

$$
\chi_1(n_0, \ldots, n_{p+1})\langle B(T_1\varphi_{n_1}^{\ell_1}, T''\Pi_{n''}U'', n_{p+1})\Pi_{n_{p+1}}u_{p+1}, \varphi_{n_0}^{\ell_0}\rangle.
\tag{3.2.18}
$$

Since $T_1\varphi_{n_1}^{\ell_1}$ is a function with values in the finite dimensional vector space G, with basis $(g_i)_i$, we decompose it as $(T_1\varphi_{n_1}^{\ell_1})^i g_i$ and write the bracket in (3.2.18) as

$$\langle a_{n',i}(x)(T_1\varphi_{n_1}^{\ell_1})^i, \varphi_{n_0}^{\ell_0}\rangle \tag{3.2.19}$$

with

$$a_{n',i}(x) = B(g_i, T''\Pi_{n''}U'', n_{p+1})\Pi_{n_{p+1}}u_{p+1}. \tag{3.2.20}$$

By (3.2.2), Sobolev injection, and the L^2 continuity of zero order pseudo-differential operators, we get for any k

$$\|\partial_x^k a_{n',i}(x)\|_{L^\infty} \leq C_k(1+|n'|)^{k+\nu+d_+}\prod_2^{p+1}\|u_j\|_{L^2} \tag{3.2.21}$$

for some fixed $\nu \in \mathbb{R}_+$. We apply estimate (2.1.2) to (3.2.19) and insert in it (3.2.21). If we use estimates of type (2.1.18) for χ_1 (replacing in (2.1.18) n_{p+1} by n_1) and the Leibniz formulas (1.2.6), (1.2.7), we see that we get for (3.2.18) estimates of type (2.1.12) as wanted.

We must now prove formula (3.2.13). Let us compute $\mathrm{Op}(b_j(u_1,\ldots,\widehat{u_j},\ldots,u_{p+1};\cdot))u_j$ using definition 2.1.2: we must in the right hand side of (3.2.16) replace V by $\mathcal{F}_{n_j}u_j$, compose on the left with $\mathcal{F}_{n_0}^*$, and sum in n_0, n_j. Using (2.1.8), we get

$$\sum_{n_0}\cdots\sum_{n_{p+1}}\chi_j(n_0,\ldots,n_{p+1})\Pi_{n_0}[b(\Pi_{n_1}u_1,\ldots,\Pi_{n_p}u_p,n_{p+1})\Pi_{n_{p+1}}u_{p+1}].$$

Consequently, because of the definition of b_χ, b_j, the operator M defined by equality (3.2.13) may be written as

$$M(u_1,\ldots,u_{p+1}) = \sum_{n_0}\cdots\sum_{n_{p+1}}\tilde{\chi}(n_0,\ldots,n_{p+1})\Pi_{n_0}[b(\Pi_{n_1}u_1,\ldots,\Pi_{n_p}u_p,n_{p+1})\Pi_{n_{p+1}}u_{p+1}] \tag{3.2.22}$$

where $\tilde{\chi}$ cuts-off outside a neighborhood of the region where one of the χ_j $j=1,\ldots,p+1$ equals one. In other words, $\tilde{\chi}$ is supported inside

$$\bigcap_{j=1}^{p+1}\{(n_0,\ldots,n_{p+1}); |n_0-n_j| \geq c(n_0+n_j) \text{ or } \exists k \in \{1,\ldots,p+1\}-\{j\} \text{ with } n_k \geq cn_0\} \tag{3.2.23}$$

for some small $c > 0$. We estimate the L^2 norm of $\Pi_{n_0}M(\Pi_{n_1}u_1,\ldots,\Pi_{n_{p+1}}u_{p+1})$ i.e. of the general term of (3.2.22). Using (3.2.3), we must bound

$$|\tilde{\chi}(n_0,\ldots,n_{p+1})|\|\Pi_{n_0}B(T_1\Pi_{n_1}u_1,\ldots,T_p\Pi_{n_p}u_p,n_{p+1})\Pi_{n_{p+1}}u_{p+1}\|_{L^2} \tag{3.2.24}$$

or equivalently the product of $|\tilde{\chi}(n_0,\ldots,n_{p+1})|$ by

$$\langle B(T_1\Pi_{n_1}u_1,\ldots,T_p\Pi_{n_p}u_p,n_{p+1})\Pi_{n_{p+1}}u_{p+1},\Pi_{n_0}u_0\rangle \tag{3.2.25}$$

for any $u_0 \in L^2$ of norm 1. If for instance n_1 and n_2 are the largest two among $n_0,\ldots,n_{p+1}$, we decompose again for $j=1,2$

$$T_j\Pi_{n_j}u_j = \sum_{\ell_j}\sum_{i_j}\langle u_j,\varphi_{n_j}^{\ell_j}\rangle(T_j\varphi_{n_j}^{\ell_j})^{i_j}g_{i_j}$$

where h^{i_j} denotes the i_jth coordinate of an element of G on the basis $(g_k)_k$. We set $n'' = (n_3, \dots, n_p)$, $n' = (n'', n_{p+1})$ and define

$$a_{i_1 i_2 n'}(x) = B(g_{i_1}, g_{i_2}, T''\Pi_{n''}U'', n_{p+1})\Pi_{n_{p+1}}u_{p+1}.$$

Then (3.2.25) may be written as the sum in ℓ_1, ℓ_2, i_1, i_2 of

$$\langle u_1, \varphi_{n_1}^{\ell_1}\rangle\langle u_2, \varphi_{n_2}^{\ell_2}\rangle\langle a_{i_1 i_2 n'}(x)(T_1\varphi_{n_1}^{\ell_1})^{i_1}(T_2\varphi_{n_2}^{\ell_2})^{i_2}, \Pi_{n_0}u_0\rangle. \tag{3.2.26}$$

The last bracket is estimated by (2.1.2). Using Sobolev injections to control the L^∞ norms of derivatives of $a_{i_1 i_2 n'}$ and (3.2.2), we may bound the modulus of (3.2.26) by

$$C\langle n_1 - n_2\rangle^{-N}(1 + n_0 + |n'|)^{\nu+N} n_{p+1}^d \prod_{\ell=0}^{p+1} \|u_\ell\|_{L^2}$$

for any N and some fixed ν. Since i_1, i_2, ℓ_1, ℓ_2 in (3.2.26) run in a finite set of indices, we get the same estimate for (3.2.25). Consequently, when the largest two among $n_0, \dots, n_{p+1}$ are among $\{n_1, \dots, n_p\}$, we have for (3.2.24) an upper bound

$$C n_{p+1}^d \frac{\mu(n_0, \dots, n_{p+1})^{\nu+N}}{S(n_0, \dots, n_{p+1})^N} \prod_1^{p+1} \|u_j\|_{L^2} \tag{3.2.27}$$

for any N. One checks in the same way that this formula holds true when one at least of the largest two among $(n_0, \dots, n_{p+1})$ equals n_0 or n_{p+1}. To conclude the proof, we have to show that estimate (3.2.27), together with the support conditions (3.2.23), implies the upper bound

$$C \frac{\max_2(n_1, \dots, n_{p+1})^{d_+ + \ell + \nu}}{\max(n_1, \dots, n_{p+1})^\ell} \frac{\mu(n_0, \dots, n_{p+1})^N}{S(n_0, \dots, n_{p+1})^N} \tag{3.2.28}$$

for any ℓ, N. If there is $c_1 > 0$ with $\max_2(n_1, \dots, n_{p+1}) \geq c_1 \max(n_1, \dots, n_{p+1})$, this is trivial. Assume now

$$\max_2(n_1, \dots, n_{p+1}) < c_1 \max(n_1, \dots, n_{p+1}).$$

If, for instance, $n_{p+1} = \max(n_1, \dots, n_{p+1})$, we have $n_{p+1} \geq \frac{1}{c_1} n_j$, $j = 1, \dots, p$. Assume moreover $|n_0 - n_{p+1}| \geq c(n_0 + n_{p+1})$ where $c > 0$ is the constant of (3.2.23). Then, if c_1 is small enough

$$S(n_0, \dots, n_{p+1}) \geq c'(n_0 + n_{p+1})$$

and inequality (3.2.27) implies (3.2.28). We are thus reduced to the case when $|n_0 - n_{p+1}| < c(n_0 + n_{p+1})$. By (3.2.23) we must have then $n_k \geq c n_0 \sim c n_{p+1}$ for some $k \in \{1, \dots, p\}$. This implies again that $\max_2(n_1, \dots, n_{p+1}) \sim \max(n_1, \dots, n_{p+1})$ and the conclusion follows. □

We shall now study symbolic properties of elements in S_p^d. To be able to get for the symbol of a composition a more explicit formula than the one of the proof of proposition 2.2.2 (ii), we shall have to limit ourselves to symbols which are "scalar" according to the following definition.

Definition 3.2.6 *Let $d \in \mathbb{R}, p \in \mathbb{N}$. We denote by $S_{p,\mathrm{sc}}^d$ the space of maps*

$$\begin{aligned} &(u_1, \dots, u_p, n_{p+1}) \to b(u_1, \dots, u_p, n_{p+1}) \\ &\mathcal{E} \times \cdots \times \mathcal{E} \times \mathbb{N}_\tau \to \mathcal{L}(\mathcal{E} \otimes \mathbb{K}, L^2(\mathbb{S}^1, G \otimes \mathbb{K})) \end{aligned} \tag{3.2.29}$$

such that there is

• A function

$$B_s : G \times \cdots \times G \times \mathbb{N}_\tau \to \mathcal{L}(G \otimes \mathbb{K}, G \otimes \mathbb{K}),\ (X_1, \ldots, X_p, n) \to B_s(X_1, \ldots, X_p, n)$$

p-linear in $(X_1, \ldots, X_p)$, satisfying for any $\alpha \in \mathbb{N}$

$$|\partial_n^\alpha B_s(X_1, \ldots, X_p, n)| \leq C_\alpha n^{d-\alpha} \prod_1^p |X_j|_G, \tag{3.2.30}$$

• A map

$$B_\infty : G \times \cdots \times G \times \mathbb{N}_\tau \to \mathcal{L}(G \otimes \mathbb{K}, G \otimes \mathbb{K}),\ (X_1, \ldots, X_p, n) \to B_\infty(X_1, \ldots, X_p, n)$$

p-linear in $(X_1, \ldots, X_p)$, such that for any $X_1, \ldots, X_p \in G$, any $n \in \mathbb{N}$, $B_\infty(X_1, \ldots, X_p, n)$ is an element of $\mathcal{L}(E_n \otimes \mathbb{K}, E_n \otimes \mathbb{K})$ whose matrix elements in the nice basis $(\varphi_n^\ell)_\ell$ of E_n satisfy for any $N \in \mathbb{N}$

$$|B_{\infty,\ell\ell'}(X_1, \ldots, X_p, n)| \leq C_N n^{-N} \prod |X_j|_G, \tag{3.2.31}$$

• A family of pseudo-differential operators of order 0 on $\mathbb{S}^1$, $T_1, \ldots, T_p$ such that one may write for any $u_1, \ldots, u_p \in \mathcal{E}, n_{p+1} \in \mathbb{N}_\tau$

$$b(u_1, \ldots, u_p, n_{p+1}) = B_s(T_1 u_1, \ldots, T_p u_p, n_{p+1}) \otimes \mathrm{Id}_{E'_{n_{p+1}} \otimes \mathbb{K}} + B_\infty(T_1 u_1, \ldots, T_p u_p, n_{p+1}). \tag{3.2.32}$$

Remark that an element of $S^d_{p,\mathrm{sc}}$ is in particular an element of S^d_p as shown by (3.2.32). In the sequel, we shall have to work with $G = \mathbb{K}^2$. In this case, B_s can be identified with a 2×2 matrix and the first term in the right hand side of (3.2.32) may be written

$$\begin{bmatrix} B_{s,11} \otimes \mathrm{Id}_{E'_{n_{p+1}} \otimes \mathbb{K}} & B_{s,12} \otimes \mathrm{Id}_{E'_{n_{p+1}} \otimes \mathbb{K}} \\ B_{s,21} \otimes \mathrm{Id}_{E'_{n_{p+1}} \otimes \mathbb{K}} & B_{s,22} \otimes \mathrm{Id}_{E'_{n_{p+1}} \otimes \mathbb{K}} \end{bmatrix} \tag{3.2.33}$$

i.e. elements of $S^d_{p,\mathrm{sc}}$ are given, up to a perturbation of order $-\infty$, by a matrix in which each block is a *scalar* operator acting on $E'_n \otimes \mathbb{K}$.

We shall use in the proof of the following proposition the fact that we can make act the scalar part of (3.2.32) not just on $E_{n_{p+1}} \otimes \mathbb{K}$ but as well on any $E_k \otimes \mathbb{K}$ (replacing $\cdot \otimes \mathrm{Id}_{E'_{n_{p+1}} \otimes \mathbb{K}}$ by $\cdot \otimes \mathrm{Id}_{E'_k \otimes \mathbb{K}}$).

Proposition 3.2.7 *(i) Let $p, q \in \mathbb{N}$. Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero, and assume that* $\operatorname{Supp} \chi$ *is small enough. There is $\nu \in \mathbb{R}$ and for any $d, d' \in \mathbb{R}$, for any symbols $a \in S^d_{q,\mathrm{sc}}, b \in S^{d'}_{p,\mathrm{sc}}$ there are a symbol $e \in \Sigma^{d+d'-1,\nu}_{p+q,1}$ and a remainder operator $M \in \mathcal{R}^{d+d',\nu}_{p+q+1}$ such that for any $U = (U', U'')$ with $U' = (u_1, \ldots, u_q) \in \mathcal{E}^q$, $U'' = (u_{q+1}, \ldots, u_{p+q}) \in \mathcal{E}^p$, any $u_{p+q+1} \in \mathcal{E}$, one has*

$$\begin{aligned} \mathrm{Op}(a_\chi(U'; \cdot))\mathrm{Op}(b_\chi(U''; \cdot))u_{p+q+1} = {} & \mathrm{Op}((a \circ b)_\chi(U; \cdot))u_{p+q+1} \\ & + \mathrm{Op}(e(U; \cdot))u_{p+q+1} \\ & + M(u_1, \ldots, u_{p+q+1}), \end{aligned} \tag{3.2.34}$$

where a_χ, b_χ are defined in terms of a, b by (3.2.5), and $a \circ b$ stands for the symbol associated to the composition $A \circ B$ of the linear maps defining a, b through (3.2.3).

(ii) Assume moreover that χ is even and that $a \in S^d_{q,\mathrm{sc}}$ satisfies $a(U';\cdot)^* = a(U';\cdot)$. Then there is a symbol $e \in \Sigma^{d-1,\nu}_{q,1}$ such that

$$\operatorname{Op}(a_\chi(U';\cdot))^* - \operatorname{Op}(a_\chi(U';\cdot)) = \operatorname{Op}(e(U';\cdot)) \tag{3.2.35}$$

for any $U' \in \mathcal{E}^q$.

Proof: (i) We decompose according to (3.2.32) $a = a_s + a_\infty$, $b = b_s + b_\infty$. Then by proposition 3.2.3, $a_{\infty,\chi}$ and $b_{\infty,\chi}$ belong to $\Sigma^{-\infty,\nu}_{p,1}$. Consequently by proposition 2.2.2, their contribution to the left hand side of (3.2.34) may be incorporated to the term e of the right hand side. In the same way, the terms $(a_\infty \circ b)_\chi$ or $(a \circ b_\infty)_\chi$ in the right hand side may be incorporated to e. We may thus assume from now on that $a = a_s$, $b = b_s$. Using notations (3.2.14), the definition (3.2.5) of a_χ, b_χ, definition 2.1.2 of quantization of a paradifferential symbol and (2.1.8), we get

$$\begin{aligned} &\operatorname{Op}(a_\chi(U';\cdot))\operatorname{Op}(b_\chi(U'';\cdot))u_{p+q+1} = \\ &\qquad \sum_{n_0} \cdots \sum_{n_{p+q+1}} \sum_k \chi_{q+1}(n_0, n', k)\chi_{p+1}(k, n'', n_{p+q+1}) \\ &\qquad\qquad \times \Pi_{n_0}\big[a(\Pi_{n'}U', k)\Pi_k[b(\Pi_{n''}U'', n_{p+q+1})\Pi_{n_{p+q+1}}u_{p+q+1}]\big] \end{aligned} \tag{3.2.36}$$

setting $n' = (n_1, \dots, n_q)$, $n'' = (n_{q+1}, \dots, n_{p+q})$. We write this expression $I + II$ where

$$\begin{aligned} I = &\sum_{n_0} \cdots \sum_{n_{p+q+1}} \sum_k \chi_{q+1}(n_0, n', k)\chi_{p+1}(k, n'', n_{p+q+1}) \\ &\qquad \times \Pi_{n_0}\big[a(\Pi_{n'}U', n_{p+q+1})\Pi_k[b(\Pi_{n''}U'', n_{p+q+1})\Pi_{n_{p+q+1}}u_{p+q+1}]\big] \end{aligned} \tag{3.2.37}$$

and

$$\begin{aligned} II = &\sum_{n_0} \cdots \sum_{n_{p+q+1}} \sum_k \chi_{q+1}(n_0, n', k)\chi_{p+1}(k, n'', n_{p+q+1}) \\ &\qquad \times \mathcal{F}^*_{n_0}\tilde{a}(\Pi_{n'}U', n_{p+q+1}; n_0, k)\tilde{b}(\Pi_{n''}U''; k, n_{p+q+1})\mathcal{F}_{n_{p+q+1}}u_{p+q+1} \end{aligned} \tag{3.2.38}$$

with

$$\begin{aligned} \tilde{a}(\Pi_{n'}U', n_{p+q+1}; n_0, k) &= \mathcal{F}_{n_0} \circ \Big[\frac{a(\Pi_{n'}U', k) - a(\Pi_{n'}U', n_{p+q+1})}{k - n_{p+q+1}}\Big] \circ \mathcal{F}^*_k \\ \tilde{b}(\Pi_{n''}U''; k, n_{p+q+1}) &= \mathcal{F}_k \circ [b(\Pi_{n''}U'', n_{p+q+1})] \circ \mathcal{F}^*_{n_{p+q+1}}(k - n_{p+q+1}). \end{aligned} \tag{3.2.39}$$

We used in the definition of I and II that a is scalar, so that in (3.2.37) it is meaningful to make act $a(\Pi_{n_{n'}}U', n_{p+q+1})$ on an element of E_k, as remarked before the statement of proposition 3.2.7.

Study of term I

We further decompose $I = I' + I''$ where

$$\begin{aligned} I' = &\sum_{n_0} \cdots \sum_{n_{p+q+1}} \chi_{p+q+1}(n_0, n', n'', n_{p+q+1}) \\ &\qquad \times \Pi_{n_0}\big[a(\Pi_{n'}U', n_{p+q+1})b(\Pi_{n''}U'', n_{p+q+1})\Pi_{n_{p+q+1}}u_{p+q+1}\big]. \end{aligned}$$

Remark that I' is nothing but the first term in the right hand side of (3.2.34). Let us show that I'' is a remainder operator. We have

$$\begin{aligned} I'' = \sum_{n_0} \cdots \sum_{n_{p+q+1}} \sum_{k} [\chi_{q+1}(n_0, n', k)\chi_{p+1}(k, n'', n_{p+q+1}) - \chi_{p+q+1}(n_0, n', n'', n_{p+q+1})] \\ \times \Pi_{n_0}\big[a(\Pi_{n'}U', n_{p+q+1})\Pi_k[b(\Pi_{n''}U'', n_{p+q+1})\Pi_{n_{p+q+1}}u_{p+q+1}]\big]. \end{aligned} \tag{3.2.40}$$

The first cut-off in the above expression is supported in a domain of form

$$\begin{aligned} &|n_0 - k| < \delta(n_0 + k), \quad |k - n_{p+q+1}| < \delta(k + n_{p+q+1}) \\ &|n'| < \delta(n_0 + k), \quad |n''| < \delta(k + n_{p+q+1}) \end{aligned} \tag{3.2.41}$$

and is equal to one on a domain of the same type. The second cut-off is supported inside a domain

$$|n_0 - n_{p+q+1}| < \delta(n_0 + n_{p+q+1}), \quad \max(|n'|, |n''|) < \delta(n_0 + n_{p+q+1}) \tag{3.2.42}$$

and is equal to 1 on a similar domain. By formula (3.2.10) of corollary 3.2.4, the general term of (3.2.40) has $\mathcal{L}(L^2, L^2)$ norm bounded from above by

$$C n_{p+q+1}^{d} \frac{|n'|^{\nu+N}}{(|n_0 - k| + |n'|)^N} n_{p+q+1}^{d'} \frac{|n''|^{\nu+N}}{(|k - n_{p+q+1}| + |n''|)^N}. \tag{3.2.43}$$

Remark moreover that by (3.2.41), (3.2.42), $n_0 \sim n_{p+q+1} \gg \max(|n'|, |n''|)$ and if

$$|n_0 - k| + |k - n_{p+q+1}| + |n'| + |n''| < \delta'(n_0 + n_{p+q+1}), \tag{3.2.44}$$

for small enough $\delta' > 0$, both cut-offs in (3.2.40) equal one. Consequently, on the support, we may always extract from one of the factors of (3.2.43) a term decaying like $(n_0 + n_{p+q+1})^{-N}$. This shows that we get for I'' remainder type estimates of form (2.1.15) with d replaced by $d + d'$.

Study of term II

We shall show that II gives the term $\mathrm{Op}(e(U; \cdot))u_{p+q+1}$ in (3.2.34). We shall need the following technical lemma:

Lemma 3.2.8 *Let $d \in \mathbb{R}$ and $f : \mathbb{Z} \to \mathbb{C}$ be a function satisfying $|\partial_n^\alpha f(n)| \le C_\alpha n^{d-\alpha}$ for any $\alpha \in \mathbb{N}$. Define for $a, b \in \mathbb{Z}, a \ne b$, $g(a,b) = \frac{f(b)-f(a)}{b-a}$. Then one may extend g to the diagonal $a = b$ and on the domain $|a - b| \le \frac{1}{2}|a + b|$ one has the estimate*

$$|\partial_a^\alpha \partial_b^\beta g(a,b)| \le C_{\alpha,\beta}(a+b)^{d-1-\alpha-\beta} \tag{3.2.45}$$

for any $\alpha, \beta \in \mathbb{N}$.

Proof: Let us construct first $\chi \in \mathcal{S}(\mathbb{R})$ real valued such that $\chi(0) = 1, \chi(n) = 0$ $\forall n \in \mathbb{Z}^*$ and, for any $k \in \mathbb{N}$, there is $\chi_k \in \mathcal{S}(\mathbb{R})$ with

$$\forall x \in \mathbb{R},\ \chi^{(k)}(x) = \partial^k \chi_k(x) \tag{3.2.46}$$

where we denote $\partial\chi(x) = \chi(x+1) - \chi(x)$ (extending notation (1.2.3) to real arguments). Take first $\gamma \in C_0^\infty(]-1,1[,\mathbb{R})$ with $\gamma(0) = 1$, $\theta \in C_0^\infty(]-\pi,\pi[,\mathbb{R})$ even, such that $\sum_{k\in\mathbb{Z}} \theta(\xi - 2\pi k) \equiv 1$. Define χ by $\hat{\chi}(\xi) = \theta(\xi)\sum_{k=-\infty}^{+\infty} \hat{\gamma}(\xi + 2k\pi)$. Then, for $n \in \mathbb{Z}$

$$\chi(n) = \frac{1}{2\pi}\int e^{in\xi}\theta(\xi)\big(\sum_{k=-\infty}^{+\infty} \hat{\gamma}(\xi + 2k\pi)\big)\,d\xi = \gamma(n).$$

Moreover

$$\chi'(x) = \frac{1}{2\pi}\int e^{ix\xi}(e^{i\xi} - 1)\hat{\chi}_1(\xi)\,d\xi = \chi_1(x+1) - \chi_1(x)$$

if we define $\hat{\chi}_1(\xi) = \frac{i\xi}{e^{i\xi}-1}\hat{\chi}(\xi)$, which belongs to $\mathcal{S}(\mathbb{R})$ by construction of $\hat{\chi}$. We deduce (3.2.46) from this equality by induction.

Write now, denoting by $\langle\cdot,\cdot\rangle$ the scalar product $\langle f_1, f_2\rangle = \sum_{-\infty}^{+\infty} f_1(n)\overline{f_2(n)}$,

$$g(a,b) = \frac{1}{b-a}\sum_{n=-\infty}^{+\infty} f(n)(\chi(n-b) - \chi(n-a)) = \langle f, H(\cdot,a,b)\rangle$$

where

$$H(n,a,b) = -\int_0^1 \chi'(n - (1-t)b - ta)\,dt.$$

This defines an extension of $g(a,b)$ to $a = b$. If we make act the finite difference operator ∂_b on $H(n,a,b)$, we get

$$\partial_b^\beta H(n,a,b) = -\int_0^1 \cdots \int_0^1 \chi^{(\beta+1)}(n - (1-t)b - ta - (s_1 + \cdots + s_\beta)(1-t))(t-1)^\beta\,ds_1\dots ds_\beta dt.$$

Using (3.2.46) in the right hand side, we see that we may write

$$\partial_b^\beta H(n,a,b) = \partial_n^{\beta+1} H_\beta(n,a,b)$$

where H_β satisfies for any $N \in \mathbb{N}$ an estimate

$$|H_\beta(n,a,b)| \le C_N \int_0^1 \langle n - (1-t)b - ta\rangle^{-N}\,dt.$$

Consequently, if we write

$$\partial_b^\beta g(a,b) = \langle f, \partial_b^\beta H(n,a,b)\rangle = \langle(\partial_n^*)^{\beta+1} f, H_\beta(n,a,b)\rangle$$

and use the above upper bound, and the assumption $|a-b| \le \frac{1}{2}|a+b|$, we obtain $|\partial_b^\beta g(a,b)| \le C|a+b|^{d-1-\beta}$. One treats in the same way the action of difference operators acting on the first variable of g. $\square$

End of proof of proposition 3.2.7: Denote by $(X_1,\dots,X_q,n) \to A(X_1,\dots,X_q,n)$ the function on $G \times \cdots \times G \times \mathbb{N}_\tau$ in terms of which the symbol $a(u_1,\dots,u_q,n)$ is defined according to definition 3.2.6 (see formula (3.2.32)). Set

$$A_1(X_1,\dots,X_q,n_{p+q+1},k) = \frac{A(X_1,\dots,X_q,n_{p+q+1}) - A(X_1,\dots,X_q,k)}{n_{p+q+1} - k}$$

(taking by convention the quotient to be the extension of lemma 3.2.8 when $n_{p+q+1} = k$). By lemma 3.2.8, A_1 satisfies when $|n_{p+q+1} - k| \leq \frac{1}{2}(n_{p+q+1} + k)$ and $n_{p+q+1} \sim k$ is large enough

$$|\partial_{n_{p+q+1}}^{\alpha_1} \partial_k^{\alpha_2} A_1(X_1, \dots, X_q, n_{p+q+1}, k)| \leq C_{\alpha\beta}(k + n_{p+q+1})^{d-1-\alpha_1-\alpha_2} \prod_{j=1}^{q} |X_j|_G.$$

In other words, assumption (3.2.11) of corollary 3.2.4 holds true. We denote by $\tilde{a}_\chi$ the product of $\tilde{a}$ given by (3.2.39) with $\chi_{q+1}(n_0, n', k)$, and by $\tilde{b}_\chi$ the product of $\tilde{b}$ by $\chi_{p+1}(k, n'', n_{p+q+1})$. By (3.2.12)

$$\begin{aligned} &\|\partial_{n_0}^{\alpha}(\partial_k^*)^{\beta_1}(\partial_{n_{p+q+1}}^*)^{\beta_2}(\partial_{n_0} - \partial_k^* - \partial_{n_{p+q+1}}^*)^{\gamma}\tilde{a}_\chi(\Pi_{n'}U', n_{p+q+1}; n_0, k)\| \\ &\qquad \leq C(n_0 + k)^{d-\beta_2-\gamma-1} \frac{|n'|^{\nu+N+\alpha+\beta_1+\beta_2+\gamma}}{(|n_0 - k| + |n'|)^N} \prod_1^q \|u_j\|_{L^2}. \end{aligned} \tag{3.2.47}$$

Moreover, by proposition 3.2.3 and Leibniz formulas (1.2.6), (1.2.7), $\tilde{b}_\chi \in \Sigma_{p,1}^{d',\nu}$ for some ν. Define now

$$e(U; n_0, n_{p+q+1}) = \sum_{n'} \sum_{n''} \sum_k \tilde{a}_\chi(\Pi_{n'}U', n_{p+q+1}; n_0, k)\tilde{b}_\chi(\Pi_{n''}U''; k, n_{p+q+1}). \tag{3.2.48}$$

By the second Leibniz formula (1.2.7)

$$\begin{aligned} (\partial_{n_0} - \partial_{n_{p+q+1}}^*)e(\Pi_{n'}U', \Pi_{n''}U''; n_0, n_{p+q+1}) &= \sum_k ((\partial_{n_0} - \partial_{n_{p+q+1}}^* - \partial_k^*)\tilde{a}_\chi)\tilde{b}_\chi \\ &\quad + \sum_k \tilde{a}_\chi(\partial_k - \partial_{n_{p+q+1}}^*)\tilde{b}_\chi \\ &\quad - \sum_k (\partial_{n_{p+q+1}}^* \tilde{a}_\chi)(\partial_{n_{p+q+1}}^* \tilde{b}_\chi). \end{aligned} \tag{3.2.49}$$

Using (3.2.47), and the fact that $\tilde{b}_\chi$ obeys symbol estimates of type (2.1.12), we see that the action of $\partial_{n_0} - \partial_{n_{p+q+1}}^*$ on e gains one unit either on the order of $\tilde{a}_\chi$ or of $\tilde{b}_\chi$ in (3.2.49), loosing a power of $|n'|$ or $|n''|$. In the same way, one sees that a ∂_{n_0} or a $\partial_{n_{p+q+1}}^*$ derivative does not change the order. Consequently, to check that $e \in \Sigma_{p+q,1}^{d+d'-1,\nu}$, we just have to check that (3.2.48) satisfies property (i) of definition 2.1.1, and estimate (2.1.12) when $\alpha = \beta = \gamma = 0$.

Since inequalities (3.2.41) are valid on the supports of $\tilde{a}_\chi$, $\tilde{b}_\chi$, (i) of definition 2.1.1 holds true (if $\delta > 0$ in (3.2.41) is small enough). Moreover, by (3.2.47) and the fact that $\tilde{b}_\chi \in \Sigma_{p,1}^{d',\nu}$, we get for $\|e(\Pi_{n'}U', \Pi_{n''}U''; n_0, n_{p+q+1})\|$ an upper bound given by

$$\sum_k (n_0 + k)^{d-1} \frac{\mu(n_0, n', k)^{\nu+N_1}}{S(n_0, n', k)^{N_1}} (k + n_{p+q+1})^{d'} \frac{\mu(k, n'', n_{p+q+1})^{\nu+N_2}}{S(k, n'', n_{p+q+1})^{N_2}} \prod_1^{p+q} \|u_j\|_{L^2}.$$

Since on the support we have $k \sim n_0 \sim n_{p+q+1}$, we may use lemma 2.2.5 to get the upper bound (for new values of ν, N)

$$C(n_0 + n_{p+q+1})^{d+d'-1} \frac{\mu(n_0, n', n'', n_{p+q+1})^{\nu+N}}{S(n_0, n', n'', n_{p+q+1})^N} \prod_1^{p+q} \|u_j\|_{L^2}$$

which is the wanted estimate.

(ii) We have using notations (2.2.1), (3.2.5) and the fact that χ is even

$$a_\chi^\bullet(U'; n_0, n_{q+1}) - a_\chi(U'; n_0, n_{q+1}) = \\ \sum_{n_1} \cdots \sum_{n_p} \chi\Big(\frac{|n'|}{n_0 + n_{q+1}}\Big)\chi\Big(\frac{n_0 - n_{q+1}}{n_0 + n_{q+1}}\Big)\mathcal{F}_{n_0} \circ \Big[a(\Pi_{n'}U', n_0) - a(\Pi_{n'}U', n_{q+1})\Big] \circ \mathcal{F}^*_{n_{q+1}}.$$

One has just to apply the proof of proposition 3.2.3 together with estimate (3.2.45) to check that the above formula defines an element of $\Sigma^{d-1,\nu}_{q,1}$. □

3.3 Polyhomogenous symbols

We collect in this subsection corollaries of the results obtained in subsections 3.1 and 3.2, which apply to symbols which are not necessarily multilinear in the arguments $u_1, \dots, u_p$.

Definition 3.3.1 *(i) For $d \in \mathbb{R}, \nu \in \mathbb{R}_+, N_0 \in \mathbb{N}^*$, we denote by $\widetilde{\Sigma}^{d,\nu}_{N_0}$ the space of functions $b : \mathcal{E} \times \mathbb{N}_\tau \times \mathbb{N}_\tau \to \mathcal{L}(\ell^2, \ell^2)$ such that there is a finite family $(b_p)_{p=0,\dots,P}$ of elements $b_p \in \Sigma^{d,\nu}_{p,N_0}$ with*

$$b(u; n_0, n_{p+1}) = \sum_{p=0}^{P} b_p(\underbrace{u, \dots, u}_{p \text{ times}}; n_0, n_{p+1}) \tag{3.3.1}$$

for any $n_0, n_{p+1} \in \mathbb{N}_\tau$, $u \in \mathcal{E}$.

(ii) For $d \in \mathbb{N}$, we denote by $\widetilde{S}^d$ the space of functions $b : \mathcal{E} \times \mathbb{N}_\tau \to \mathcal{L}(\mathcal{E}, L^2)$ such that there is a finite family $(b_p)_{p=0,\dots,P}$ of elements $b_p \in S^d_p$ with

$$b(u, n) = \sum_{p=0}^{P} b_p(\underbrace{u, \dots, u}_{p \text{ times}}, n) \tag{3.3.2}$$

for any $n \in \mathbb{N}_\tau, u \in \mathcal{E}$. We define in a similar way $\widetilde{S}^d_{\mathrm{sc}}$ from $S^d_{p,\mathrm{sc}}$.

(iii) For $\nu \in \mathbb{R}_+, d \in \mathbb{R}$, we denote by $\widetilde{\mathcal{R}}^{d,\nu}$ the space of all maps $M : \mathcal{E} \to L^2$ such that there is a finite family of maps $M_p \in \mathcal{R}^{d,\nu}_p$ $p = 1, \dots, P$ with

$$M(u) = \sum_{p=1}^{P} M_p(\underbrace{u, \dots, u}_{p \text{ times}}) \tag{3.3.3}$$

for any $u \in \mathcal{E}$. Some times, we shall use the same notation for maps $(u, v) \to M(u, v)$ depending on two arguments $u, v \in \mathcal{E}$, and which may be written as a sum of multilinear expressions of form $M_p(u, \dots, u, v \dots, v)$ where the total number of arguments is p and $1 \le p \le P$.

We define the valuation $v(b)$ of a symbol b (resp. $v(M)$ of an element M of $\widetilde{\mathcal{R}}^{d,\nu}$) as the smallest $p \geq 0$ (resp. $p \geq 1$) such that $b_p \not\equiv 0$ in (3.3.1), (3.3.2) (resp. $M_p \not\equiv 0$ in (3.3.3)). The modified valuation $v'(b)$ of a symbol is the smallest $p \geq 1$ such that $b_p \not\equiv 0$ in (3.3.1), (3.3.2).

In section 4 below, we shall have to use symbols verifying conditions of type (1.1.3). We introduce the following definition.

Definition 3.3.2 *Let κ be an odd integer, $r \in \mathbb{N}$ with $\kappa \leq r-1 \leq 2\kappa$. We say that a symbol $b \in \widetilde{\Sigma}^{d,\nu}_{N_0}$ (resp. $b \in \widetilde{S}^d$, resp. an operator $M \in \widetilde{\mathcal{R}}^{d,\nu}$) satisfies condition $C(\kappa, r)$ if and only if $b = b_0 + \sum_{p=\kappa}^{\kappa_1} b_p$ (resp. $M = \sum_{p=\kappa}^{\kappa_1} M_{p+1}$) with $b_p \in \Sigma^{d,\nu}_{p,N_0}$ (resp. $b_p \in S^d_p$, resp. $M_{p+1} \in \mathcal{R}^{d,\nu}_{p+1}$) and $b_p \equiv 0$ (resp. $M_{p+1} \equiv 0$) when p is an even integer $2k$ satisfying $\kappa \leq 2k < r-1$.*

We shall use below several times the following remark. Let L be a linear map (resp. B be a bilinear map) from one (resp. the product of two) of the above spaces of symbols or operators to a third space of that type. Assume that L (resp. B) respects the natural graduations of these spaces. Then L (resp. B) sends symbols or operators satisfying $C(\kappa, r)$ to symbols or operators satisfying $C(\kappa, r)$.

This is trivial for linear maps. In the bilinear case, this follows from the fact that in an expression of form $B(a, b)$, the contributions of type $B(a_q, b_p)$ with $q > 0$ and $p > 0$ are homogeneous of degree $p + q \geq 2\kappa \geq r - 1$ (since $v'(a) \geq \kappa$, $v'(b) \geq \kappa$), so the condition imposed by $C(\kappa, r)$ on $B(a_q, b_p)$ is void. Only terms of type $B(a_0, b_p)$, $B(a_q, b_0)$ have to be taken into consideration, and they satisfy the condition of the definition.

We extend the definition of the quantization of operators by linearity, setting for $b \in \widetilde{\Sigma}^{d,\nu}_{N_0}$ or $b \in \widetilde{S}^d$ respectively

$$
\begin{aligned}
\operatorname{Op}(b(u;\cdot)) &= \sum_{p=0}^{P} \operatorname{Op}(b_p(u,\dots,u;\cdot)) \\
\widetilde{\operatorname{Op}}(b(u,\cdot)) &= \sum_{p=0}^{P} \widetilde{\operatorname{Op}}(b_p(u,\dots,u,\cdot)).
\end{aligned}
\tag{3.3.4}
$$

By proposition 2.1.3 and lemma 2.1.7, maps like $(u, v) \to \operatorname{Op}(b(u;\cdot))v$, for $b \in \widetilde{\Sigma}^{d,\nu}_{N_0}$ (resp. $u \to R(u)$ for $R \in \widetilde{\mathcal{R}}^{d,\nu}$) extend from $\mathcal{E} \times \mathcal{E}$ (resp. $\mathcal{E}$) to $H^s(\mathbb{S}^1, G \otimes \mathbb{K})^2$ (resp. $H^s(\mathbb{S}^1, G \otimes \mathbb{K})$) if s is large enough. We use this in the following corollaries, which are stated for arguments u, v smooth enough, but need only to be checked when $u, v \in \mathcal{E}$ by density.

Corollary 3.3.3 *Let $P \in \mathbb{N}^*$ be given. There is $\nu \in \mathbb{R}_+$ such that if we define for $d \in \mathbb{R}$, $b \in \widetilde{S}^d, \chi \in C_0^\infty(\mathbb{R}), \chi \equiv 1$ close to zero,* $\operatorname{Supp}\chi$ *small enough, $b_\chi = \sum_{p=0}^{P} b_{p,\chi} \in \widetilde{\Sigma}^{d,\nu}_1$, we may find a symbol $b^0 \in \widetilde{\Sigma}^{0,\nu+d_+}_1$ and an operator $M \in \widetilde{\mathcal{R}}^{0,\nu+d_+}$ such that for any smooth enough u*

$$
\widetilde{\operatorname{Op}}(b(u,\cdot))u = \operatorname{Op}(b_\chi(u;\cdot))u + \operatorname{Op}(b^0(u;\cdot))u + M(u).
\tag{3.3.5}
$$

Moreover, one has

$$
v(b_\chi) \geq v(b),\ v(b^0) \geq v'(b),\ v(M) \geq v'(b) + 1
\tag{3.3.6}
$$

and b_χ, b^0, M satisfy condition $C(\kappa, r)$ if b does so.

Proof: We decompose $b = \sum_{p=0}^{P} b_p$ and apply to each component proposition 3.2.5. We obtain (3.3.5) and (3.3.6), remembering that for $p = 0$, b_0 does not depend on u, so that $\widetilde{\mathrm{Op}}(b_0) = \mathrm{Op}(b_{0,\chi})$ for any χ as in the statement of the theorem. Consequently b_0 does not contribute to the last two terms in (3.3.5), which implies the last two inequalities in (3.3.6). $\square$

Corollary 3.3.4 *Let $d, d' \in \mathbb{R}, a \in \widetilde{S}^{d}_{\mathrm{sc}}, b \in \widetilde{S}^{d'}_{\mathrm{sc}}$. Let $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to zero, with small enough support. There are $\nu \in \mathbb{R}_+$ independent of d, d', a symbol $e \in \widetilde{\Sigma}_1^{d+d'-1,\nu}$ and a remainder operator $M \in \widetilde{\mathcal{R}}^{d+d',\nu} \subset \widetilde{\mathcal{R}}^{0,\nu+d_+ + d'_+}$ such that for any smooth enough u, v,*

$$\mathrm{Op}(a_\chi(u;\cdot)) \circ \mathrm{Op}(b_\chi(u;\cdot))v = \mathrm{Op}((a \circ b)_\chi(u;\cdot))v + \mathrm{Op}(e(u;\cdot))v + M(u,v). \tag{3.3.7}$$

Moreover

$$v(e) \geq \min(v'(a), v'(b)), \ v(M) \geq \min(v'(a), v'(b)) + 1. \tag{3.3.8}$$

If $v(a) = v'(a) > 0$, $v(b) = v'(b) > 0$, we have

$$v(e) \geq v'(a) + v'(b), \ v(M) \geq v'(a) + v'(b) + 1. \tag{3.3.9}$$

Moreover $a \circ b$, e and M satisfy $C(\kappa, r)$ if a and b do so.

Proof: We decompose $a = \sum_{q=0}^{Q} a_q$, $b = \sum_{p=0}^{P} b_p$ and apply proposition 3.2.7 to each contribution, remarking that $\mathrm{Op}(a_{0,\chi})\mathrm{Op}(b_{0,\chi}) = \mathrm{Op}((a_0 \circ b_0)_\chi)$, so that all contributions to e and M come from compositions with $p > 0$ or $q > 0$. The last statement comes from the remark after definition 3.3.2. $\square$

Corollary 3.3.5 *(i) Let $\nu \in \mathbb{R}_+, N_0 \in \mathbb{N}^*$. There is $\nu' \in \mathbb{R}_+$, and for any $d, d' \in \mathbb{R}$, any $a \in \widetilde{\Sigma}^{d,\nu}_{N_0}$, $b \in \widetilde{\Sigma}^{d',\nu}_{N_0}$ satisfying condition $(i)_\delta$ of definition 2.1.1 with small enough $\delta > 0$, there is a symbol $a\#b \in \widetilde{\Sigma}^{d+d',\nu'}_{N_0}$ such that for any smooth enough u*

$$\mathrm{Op}(a(u;\cdot)) \circ \mathrm{Op}(b(u;\cdot))u = \mathrm{Op}(a\#b(u;\cdot))u.$$

Moreover $v(a\#b) \geq v(a) + v(b)$, and $a\#b$ satisfies $C(\kappa, r)$ if a, b do so.

(ii) Assume moreover that the homogeneous components $a_q(u; n_0, n_{q+1})$ and $b_p(u; n'_0, n'_{p+1})$ of a, b commute for large enough $n_0, n_{q+1}, n'_0, n'_{p+1}$ and that $a_0 b_0 \equiv b_0 a_0$. There is $c \in \widetilde{\Sigma}^{d+d'-1,\nu'}_{N_0}$ such that

$$[\mathrm{Op}(a(u;\cdot)), \mathrm{Op}(b(u;\cdot))]u = \mathrm{Op}(c(u;\cdot))u$$

for any smooth enough u, and $v(c) \geq \min(v'(a), v'(b))$. If moreover $v(a) = v'(a) > 0$ and $v(b) = v'(b) > 0$, then $v(c) \geq v'(a) + v'(b)$. Finally if a, b satisfy $C(\kappa, r)$, the same holds true for c.

Proof: We decompose again $a = \sum_{p=0}^{P} a_p, b = \sum_{q=0}^{Q} b_q$ and define $a\#b$ or c using the linearity in (i), (ii) of proposition 2.2.2. The statement concerning valuations in (ii) of the corollary comes from the fact that $[\mathrm{Op}(a_0), \mathrm{Op}(b_0)] = 0$ since these operators are constant coefficient ones. $\Box$

Corollary 3.3.6 *(i) Let $d, d' \in \mathbb{R}, p \in \mathbb{N}^*, \nu \in \mathbb{R}_+, N_0 \in \mathbb{N}^*$. Let $a \in \Sigma^{d,\nu}_{p,N_0}$, $b \in \widetilde{\Sigma}^{d',\nu}_{N_0}$, and assume that they satisfy condition $(i)_\delta$ of definition 2.1.1 with a small enough $\delta > 0$. Then there are $\nu' = 2\nu + d'_+ + 1$ and $c \in \widetilde{\Sigma}^{d,\nu'}_{N_0}$ with*

$$\mathrm{Op}[a[\mathrm{Op}(b(u;\cdot))u, \underbrace{u, \dots, u}_{p-1 \text{ times}}; \cdot]]v = \mathrm{Op}(c(u;\cdot))v$$

for any smooth enough u, v. Moreover $v(c) \geq p + v(b)$ and c satisfies $C(\kappa, r)$ if b does so and p is odd, $p \geq \kappa$.

(ii) Let $d \in \mathbb{R}$, $d' \in \mathbb{R}$, $\nu \in \mathbb{R}_+, p \in \mathbb{N}^*, N_0 \in \mathbb{N}^*, a \in \Sigma^{d,\nu}_{p,N_0}$ satisfying condition $(i)_\delta$ of definition 2.1.1 with a small enough $\delta > 0$. Let $M \in \widetilde{\mathcal{R}}^{d',\nu}$. There are $\nu' = d'_+ + 2\nu + 1$, $\nu'' = 2\nu + 1$, $b \in \widetilde{\Sigma}^{d,\nu'}_{N_0}$ and $R \in \widetilde{\mathcal{R}}^{d+d'_+,\nu''} \subset \widetilde{\mathcal{R}}^{0,\nu''+d_++d'_+}$ such that for any smooth enough u, v

$$\mathrm{Op}(a(M(u), \underbrace{u, \dots, u}_{p-1}; \cdot))v = \mathrm{Op}(b(u;\cdot))v + R(u,v)$$

with $v(b) \geq v(M) + p - 1$, $v(R) \geq v(M) + p$. Moreover b, R satisfy $C(\kappa, r)$ if M does so and p is odd, $p \geq \kappa$.

(iii) Let $d \in \mathbb{R}, d' \in \mathbb{R}$, $\nu, \nu' \in \mathbb{R}_+, N_0 \in \mathbb{N}^*, a \in \widetilde{\Sigma}^{d,\nu}_{N_0}, M \in \widetilde{\mathcal{R}}^{d',\nu}$. There is $\nu'' = \nu + \nu' + 1$ such that $u \to R(u) = \mathrm{Op}(a(u;\cdot))M(u)$ is in $\widetilde{\mathcal{R}}^{d+d',\nu''}$ and $v(R) \geq v(a) + v(M)$. Moreover R satisfies $C(\kappa, r)$ if a and M do so.

These statements follow from propositions 2.2.3, 2.2.4 and 2.2.6. In the same way, we deduce from proposition 2.2.7:

Corollary 3.3.7 *(i) Let $d, d' \in \mathbb{R}, \nu, \nu' \in \mathbb{R}_+, N_0 \in \mathbb{N}, q \in \mathbb{N}^*$. Let $a \in \widetilde{\Sigma}^{d,\nu}_{N_0}$ and $M \in \mathcal{R}^{d',\nu'}_q$. There is $\nu'' = d_+ + \nu + \nu' + 1$ such that the operator $u \to R(u)$ given by $R(u) = M(\mathrm{Op}(a(u;\cdot))u, u, \dots, u)$ is in $\widetilde{\mathcal{R}}^{d',\nu''}$ with $v(R) \geq q + v(a)$. Moreover R satisfies $C(\kappa, r)$ if a does so and $q - 1$ is an odd integer $q - 1 \geq \kappa$.*

(ii) Let $M_1 \in \mathcal{R}^{d,\nu}_q, M_2 \in \widetilde{\mathcal{R}}^{d',\nu'}$. Then there is $\nu'' = \nu + \nu' + d'_+ + 1$ such that $R(u) = M_1(M_2(u), u, \dots, u)$ is in $\widetilde{\mathcal{R}}^{d,\nu''}$ with $v(R) \geq v(M_2) + q - 1$.

Let us conclude this subsection with the following technical lemma.

Lemma 3.3.8 *(i) Let $a(\lambda)$ be a smooth function on $\mathbb{R}_+$ satisfying for any k, $|\partial^k_\lambda a(\lambda)| \leq C_k \lambda^{1-k}$ when $\lambda \to +\infty$. Let $(n_1, \dots, n_p) = n' \to G(n')$ be a real valued function defined on $\mathbb{N}^p_\tau$, such that there is $C > 0$ with $|G(n')| \leq C(1 + |n'|)$. Consider the function*

$$F(n_0, n_1, \dots, n_{p+1}) = a(n_0) - a(n_{p+1}) + G(n_1, \dots, n_p) \tag{3.3.10}$$

and assume that there is $c > 0, N_0 \in \mathbb{N}^*$ such that for any $n_0, n_{p+1} \in \mathbb{N}_\tau, n' \in \mathbb{N}_\tau^p$ satisfying $|n_0 - n_{p+1}| \leq \frac{1}{4}(n_0 + n_{p+1})$, $|n'| \leq \frac{1}{4}(n_0 + n_{p+1})$ one has

$$|F(n_0, \dots, n_{p+1})| \geq c(1 + |n_0 - n_{p+1}|)|n'|^{-N_0}. \tag{3.3.11}$$

Then we have for any $\alpha, \beta, \gamma \in \mathbb{N}$, any $(n_0, n', n_{p+1}) \in \mathbb{N}_\tau^{p+2}$ satisfying the preceding inequalities

$$\begin{aligned}&\Big|\partial_{n_0}^\alpha (\partial_{n_{p+1}}^*)^\beta (\partial_{n_0} - \partial_{n_{p+1}}^*)^\gamma \frac{1}{F(n_0, \dots, n_{p+1})}\Big| \\ &\qquad \leq C_{\alpha\beta\gamma}(n_0 + n_{p+1})^{-\gamma} |n'|^{N_0(\alpha+\beta+\gamma+1)} (1 + |n_0 - n_{p+1}|)^{-1}.\end{aligned} \tag{3.3.12}$$

(ii) If instead of (3.3.11), F satisfies when $|n_0 - n_{p+1}| \leq \frac{1}{4}(n_0 + n_{p+1})$, $|n'| \leq \frac{1}{4}(n_0 + n_{p+1})$

$$|F(n_0, \dots, n_{p+1})| \geq c(n_0 + n_{p+1})|n'|^{-N_0}, \tag{3.3.13}$$

then (3.3.12) holds true with the right hand side replaced by

$$C_{\alpha\beta\gamma}(n_0 + n_{p+1})^{-1-\gamma} |n'|^{N_0(\alpha+\beta+\gamma+1)}. \tag{3.3.14}$$

Proof: (i) We may assume in (3.3.12) that $\alpha+\beta+\gamma > 0$ since the inequality without derivatives follows from (3.3.11). Remark that we have then

$$|\partial_{n_0}^\alpha (\partial_{n_{p+1}}^*)^\beta (\partial_{n_0} - \partial_{n_{p+1}}^*)^\gamma F(n_0, \dots, n_{p+1})| \leq C(1 + |n_0 - n_{p+1}|)(n_0 + n_{p+1})^{-\gamma}. \tag{3.3.15}$$

This follows from lemma 3.2.8 applied to $g(n_0, n_{p+1}) = \frac{a(n_0) - a(n_{p+1})}{n_0 - n_{p+1}}$ and from Leibniz formulas (1.2.6), (1.2.7). We shall show that for any α, β, γ we may write the quantity estimated in the left hand side of (3.3.12) as a linear combination, indexed by $k = 1, \dots, \alpha+\beta+\gamma$, of expressions of form

$$\frac{H_k}{F_1 \cdots F_{k+1}}(n_0, \dots, n_{p+1}), \tag{3.3.16}$$

where each function H_k satisfies

$$|\partial_{n_0}^{\alpha'} (\partial_{n_{p+1}}^*)^{\beta'} (\partial_{n_0} - \partial_{n_{p+1}}^*)^{\gamma'} H_k(n_0, \dots, n_{p+1})| \leq C(1 + |n_0 - n_{p+1}|)^k (n_0 + n_{p+1})^{-\gamma-\gamma'}, \tag{3.3.17}$$

and where $F_1, \dots, F_{k+1}$ verify (3.3.11). Inequality (3.3.12) will then follow from (3.3.17) with $\alpha' = \beta' = \gamma' = 0$.

To obtain the structure (3.3.16), we just have to show that if we apply to (3.3.16) a derivative $\partial_{n_0}^{\alpha_0} (\partial_{n_{p+1}}^*)^{\beta_0} (\partial_{n_0} - \partial_{n_{p+1}}^*)^{\gamma_0}$ with $\alpha_0 + \beta_0 + \gamma_0 = 1$, we get the sum of an expression $\tilde{H}_k (F_1 \cdots F_{k+1})^{-1}$, where $\tilde{H}_k$ satisfies (3.3.17) with γ replaced by $\gamma + \gamma_0$, and of a quantity $\tilde{H}_{k+1}(\tilde{F}_1 \cdots \tilde{F}_{k+2})^{-1}$, with $\tilde{H}_{k+1}$ satisfying (3.3.17) with k replaced by $k+1$ and γ by $\gamma+\gamma_0$, and with $\tilde{F}_j$ verifying (3.3.11). This follows from Leibniz formulas (1.2.6), (1.2.7) and from (3.3.15), (3.3.17). This concludes the proof.

(ii) The proof is the same, replacing in (3.3.15), (3.3.17) the $1 + |n_0 - n_{p+1}|$ factor by $n_0 + n_{p+1}$. □

4 Long time existence

4.1 Strategy of proof

The aim of this section is to prove theorem 1.1.1. Our strategy will be to combine the methods used by Bourgain [5], Bambusi [1], Bambusi and Grébert [3], Delort and Szeftel [10] for semi-linear equations, with the well-known approach allowing one to obtain quasi-linear energy inequalities, namely diagonalization of the principal symbol of the equation.

Let us describe the steps that we shall follow, forgetting the necessary technicalities we shall have to introduce later on. We denote by $\Lambda_m = \sqrt{-\Delta + V + m^2}$, and we shall consider an equivalent system to the scalar equation for $u = \begin{bmatrix} \Lambda_m v \\ \partial_t v \end{bmatrix}$, of type $\partial_t u = \widetilde{\mathrm{Op}}(M(u,\cdot))u$, where M will be a symbol of order 1, belonging to the class introduced in subsection 3.1. We would like to control over long time intervals the Sobolev energy of u

$$\langle \Lambda_m^s u(t,\cdot), \Lambda_m^s u(t,\cdot)\rangle. \tag{4.1.1}$$

If one computes the time derivative of this expression, one gets

$$2\mathrm{Re}\,\langle \Lambda_m^s \widetilde{\mathrm{Op}}(M(u,\cdot))u, \Lambda_m^s u\rangle. \tag{4.1.2}$$

If $M(u,\cdot) = M_0(\cdot) + M^\kappa(u,\cdot)$ is the sum of two anti-self-adjoint matrices, with M_0 independent of u and M^κ homogeneous of degree $\kappa > 0$ in u, symbolic calculus shows that the above expression may be written as

$$\langle \widetilde{\mathrm{Op}}(b(u,\cdot))u, u\rangle \tag{4.1.3}$$

where b is a self-adjoint symbol of order $2s$ vanishing at least at order κ at $u = 0$. Consequently, for s large enough, this bracket is bounded from above by $C\|u\|_{H^s}^{\kappa+2}$, and one gets the estimate

$$\frac{d}{dt}\|u(t,\cdot)\|_{H^s}^2 \leq C\|u(t,\cdot)\|_{H^s}^{\kappa+2}. \tag{4.1.4}$$

This is a way to recover the local existence result asserting that for smooth data of size $\epsilon \to 0$, the solution exists at least over an interval of time of length $c\epsilon^{-\kappa}$. Our goal here is to obtain a better result when κ is odd (and when the parameter m is outside a subset of zero measure). Namely we want to obtain a solution over a time interval of length $c\epsilon^{-2\kappa}$. From (4.1.1) to (4.1.3) we know that

$$\frac{d}{dt}\langle \Lambda_m^s u(t,\cdot), \Lambda_m^s u(t,\cdot)\rangle = \langle \widetilde{\mathrm{Op}}(b(u,\cdot))u, u\rangle. \tag{4.1.5}$$

We would like to add in the left hand side a new contribution, of form $\langle \widetilde{\mathrm{Op}}(a(u,\cdot))u, u\rangle$, vanishing at order $\kappa+2$ at 0, with a symbol a of order $2s$, determined in such a way that the time derivative of this quantity will cancel out the right hand side of (4.1.5), up to remainders $O(\|u\|_{H^s}^{2\kappa+2})$. If we compute $\frac{d}{dt}\langle \widetilde{\mathrm{Op}}(a(u,\cdot))u, u\rangle$ we get from the action of d/dt on the u's which are not in the argument of a, a contribution of type

$$\langle [\widetilde{\mathrm{Op}}(a(u,\cdot))\widetilde{\mathrm{Op}}(M(u,\cdot)) + \widetilde{\mathrm{Op}}(M(u,\cdot))^*\widetilde{\mathrm{Op}}(a(u,\cdot))]u, u\rangle. \tag{4.1.6}$$

Remember that $M(u,\cdot) = M_0(\cdot) + M^\kappa(u,\cdot)$. Consider the expression obtained replacing in (4.1.6) $M(u,\cdot)$ by $M^\kappa(u,\cdot)$: we get a term homogeneous of degree $2\kappa+2$ in u. In a semi-linear framework, i.e. when M^κ is a symbol of order 0, this gives a contribution to (4.1.6) which is $O(\|u\|_{H^s}^{2\kappa+2})$, since a is of order $2s$. In our quasi-linear framework, $M^\kappa(u,\cdot)$ is a symbol of order 1, which *a priori* looses one extra derivative. The way to circumvent that difficulty is well known: one has to arrange so that a be self-adjoint and commute to M^κ. Then since $M^\kappa(u,\cdot)$ is assumed anti-self-adjoint, the contribution of M^κ to (4.1.6) may be written in terms of a commutator $[\widetilde{\mathrm{Op}}(a(u,\cdot)), \widetilde{\mathrm{Op}}(M^\kappa(u,\cdot))]u$. The symbolic calculus we studied in the preceding sections shows that this commutator gains one derivative, so that again the contribution of M^κ to (4.1.6) is $O(\|u\|_{H^s}^{2\kappa+2})$. In other words, up to such nice remainders, $\frac{d}{dt}\langle \widetilde{\mathrm{Op}}(a(u,\cdot))u, u\rangle$ will be given by contributions of type (4.1.6) with M replaced by M_0, and by similar terms coming from the action of $\frac{d}{dt}$ on those u inside the argument of a. The last step of the proof will be to show that we may choose a so that these contributions to $\frac{d}{dt}\langle \widetilde{\mathrm{Op}}(a(u,\cdot))u, u\rangle$ will cancel out the right hand side of (4.1.5).

To ensure the commutator property of a with M, we start instead of (4.1.1) with

$$\langle \Lambda_m^s \tilde{u}(t,\cdot), \Lambda_m^s \tilde{u}(t,\cdot)\rangle \tag{4.1.7}$$

where $\tilde{u}$ is a new unknown defined in terms of u by $\tilde{u} = Q(u)u$, Q being a matrix such that $D(u,\cdot) = Q(u)M(u,\cdot)Q(u)^{-1}$ is diagonal. Computing the time derivative of (4.1.7), we shall get instead of (4.1.5) an expression

$$\langle \widetilde{\mathrm{Op}}(b(u,\cdot))\tilde{u}, \tilde{u}\rangle \tag{4.1.8}$$

that we will try to cancel out adding to (4.1.7) a quantity

$$\langle \widetilde{\mathrm{Op}}(a(u,\cdot))\tilde{u}, \tilde{u}\rangle \tag{4.1.9}$$

where a is again a symbol to be determined. When we shall compute the time derivative of (4.1.9), the contribution corresponding to (4.1.6) will be

$$\langle [\widetilde{\mathrm{Op}}(a(u,\cdot))\widetilde{\mathrm{Op}}(D(u,\cdot)) + \widetilde{\mathrm{Op}}(D(u,\cdot))^*\widetilde{\mathrm{Op}}(a(u,\cdot))]\tilde{u}, \tilde{u}\rangle. \tag{4.1.10}$$

Since now D is diagonal, and since we shall look for a diagonal symbol a, the commutation property between symbols $aD = Da$ will hold true automatically. Moreover a will be taken self-adjoint and D will be anti-self-adjoint. Because of that, the contribution of the part of D which is homogeneous in u of order κ to (4.1.10) will be expressed through a commutator, and will provide a remainder of type $\|u\|_{H^s}^{2\kappa+2}$. As explained above, the terms coming from the part D_0 of D independent of u will cancel out (4.1.8), if the symbol a is conveniently defined in terms of b. Finally, since for small functions u, (4.1.7) will be equivalent to $\|u(t,\cdot)\|_{H^s}^2$ we shall get

$$\frac{d}{dt}\|u(t,\cdot)\|_{H^s}^2 \le C\|u(t,\cdot)\|_{H^s}^{2\kappa+2}$$

as long as $\|u(t,\cdot)\|_{H^s}$ stays small enough, which is what we need to get a solution defined on an interval of length $c\epsilon^{-2\kappa}$.

Let us mention that the computations we outlined above will have to be done using paradifferential operators instead of pseudo-differential ones. This is the justification for our study of the former in section 2. The diagonalization of the principal symbol of the equation, i.e. the construction of $\tilde{u}$ in terms of u, will be described in subsection 4.2. The last subsection 4.3 will be devoted to the construction of the correcting terms (4.1.9) and to the proof of the theorem.

4.2 Diagonalization of principal part

We shall denote by $\Lambda_m = \sqrt{-\Delta + V + m^2}$. This is a scalar invertible pseudo-differential operator of order 1 on $\mathbb{S}^1$. If $v \in H^{s+1}(\mathbb{S}^1, \mathbb{R})$ for a large enough s, we set

$$u = \begin{bmatrix} \Lambda_m v \\ \partial_t v \end{bmatrix}, v = \Lambda_m^{-1} u_1,\ \partial_t v = u_2. \tag{4.2.1}$$

We define

$$a(u) = c(\Lambda_m^{-1} u_1, u_2, \partial_x \Lambda_m^{-1} u_1) \tag{4.2.2}$$

where c is the function defined in (1.1.1), (1.1.2). In particular, $a(u)$ may be written as a sum of multilinear expressions in Tu_1, u_2 for pseudo-differential operators of order 0, T. Consequently $a(u)$ will be, according to definitions 3.3.1 and 3.2.1, a symbol of $\widetilde{S}^0_{\text{sc}}$ (independent of n). Its valuation will be equal to κ which, according to assumption (1.1.3), may be assumed to be odd. Moreover it will satisfy condition $C(\kappa, r)$ of definition 3.3.2 i.e.

$$a = \sum_{k=\kappa}^{\kappa_1} a_k(u) \text{ where } a_k \in S^0_k, a_{2k} \equiv 0 \text{ for } \kappa \leq 2k < r - 1. \tag{4.2.3}$$

The first equation of (1.1.4) may be written

$$\partial_t u = \begin{bmatrix} 0 & \Lambda_m \\ -(1 + a(u))^2 \Lambda_m & 0 \end{bmatrix} u. \tag{4.2.4}$$

We shall denote by G the vector space $\mathbb{R}^2$, and consider the operator $-\frac{d^2}{dx^2} + V(x)$ acting on $L^2(\mathbb{S}^1, G)$. As in section 2.1, we denote by $(\omega_n^-)^2 \leq (\omega_n^+)^2$ the couple of eigenvalues with asymptotics (1.2.1), and by Π_n the spectral projector on the subspace of $L^2(\mathbb{S}^1, G)$ generated by the eigenfunctions associated to these two eigenvalues for $n \geq \tau + 1$ large enough. We denote by E_n the range of Π_n. Then E_n is four dimensional for $n \geq \tau + 1$. We define E_τ to be the orthogonal complement in $L^2(\mathbb{S}^1, G)$ of the Hilbert sum $\bigoplus_{n \geq \tau+1} E_n$. Then E_τ is even dimensional and we have the Hilbert decomposition

$$L^2(\mathbb{S}^1, G) = \bigoplus_{n=\tau}^{+\infty} E_n. \tag{4.2.5}$$

At times we shall denote by $E'_n, n \geq \tau + 1$ the subspace of $L^2(\mathbb{S}^1, \mathbb{R})$ generated by the two eigenfunctions associated to the eigenvalues $(\omega_n^-)^2$ and $(\omega_n^+)^2$ of the operator $-\frac{d^2}{dx^2} + V(x)$ acting on $L^2(\mathbb{S}^1, \mathbb{R})$. We define E'_τ in a similar way as E_τ. We have for $n \geq \tau$, $E_n \simeq E'_n \times E'_n$. We denote by $\mathcal{E}$ the algebraic direct sum of E_n for $n \geq \tau$. We introduce the following matrices

$$P(u, n) = \begin{bmatrix} \mathrm{I}_{K'(n)} & \mathrm{I}_{K'(n)} \\ i(1 + a(u))\mathrm{I}_{K'(n)} & -i(1 + a(u))\mathrm{I}_{K'(n)} \end{bmatrix} \tag{4.2.6}$$

and

$$Q(u,n) = \frac{i}{2}\begin{bmatrix} -i(1+a(u))\mathrm{I}_{K'(n)} & -\mathrm{I}_{K'(n)} \\ -i(1+a(u))\mathrm{I}_{K'(n)} & \mathrm{I}_{K'(n)} \end{bmatrix} \tag{4.2.7}$$

so that

$$P(u,n)Q(u,n) = Q(u,n)P(u,n) = (1+a(u))\mathrm{I}_{2K'(n)} \tag{4.2.8}$$

where $K'(n) = \dim E'_n = 2$ when $n > \tau$. (We prefer to use $Q(u,n)$ instead of $P(u,n)^{-1}$ to always work with matrices whose coefficients are polynomial in u). Then, according to definitions 3.3.1 and 3.2.1, P and Q are elements of $\widetilde{S}^0$. Actually these matrices define, according to definition 3.2.6 and (3.2.33) elements of $\widetilde{S}^0_{\mathrm{sc}}$, since each block of $P(u,n)$, $Q(u,n)$ is a scalar matrix (the contribution of order $-\infty$ of definition 3.2.6 is zero in this case). Moreover

$$v(P) = v(Q) = 0,\ v'(P) = v'(Q) = \kappa \tag{4.2.9}$$

and $P(u,n)$ and $Q(u,n)$ satisfy condition $C(\kappa, r)$.

Remind that we have constructed in theorem 1.2.1 a nice basis of $L^2(\mathbb{S}^1,\mathbb{R})$, which was adapted to the decomposition given by the E'_n (which were then denoted by E_n). We construct from this nice basis a natural basis of $E_n = E'_n \times E'_n$, which makes a nice basis of $L^2(\mathbb{S}^1, G)$, as at the beginning of subsection 3.2. We denote by $\lambda_m(n)$ the matrix of $\Lambda_m|_{E'_n}$ in the above nice basis. For $n \geq \tau+1$, $\lambda_m(n)$ is a 2×2 matrix. We denote by $\omega(\lambda)$ a symbol of order 1 on $\mathbb{R}_+$ with asymptotics given by (1.2.1) and we define

$$\omega_m(n) = \sqrt{m^2 + \omega(n)^2} \tag{4.2.10}$$

so that the difference between the eigenvalues of $\sqrt{-\Delta + V + m^2}|_{E'_n}$ and $\omega_m(n)$ is $O(n^{-\infty})$ when $n \to +\infty$. The matrix $\lambda_m(n)$ may be written

$$\lambda_m(n) = \omega_m(n)\mathrm{I}_{K'(n)} + \hat{\lambda}_m(n) \tag{4.2.11}$$

where $\hat{\lambda}_m(n)$ is a matrix whose norm decays like $n^{-\infty}$ when $n \to +\infty$. We introduce for $n \geq \tau$ the matrix

$$M(u,n) = \begin{bmatrix} 0 & \lambda_m(n) \\ -(1+a(u))^2\lambda_m(n) & 0 \end{bmatrix}. \tag{4.2.12}$$

This is a $K(n)\times K(n)$ matrix (where $K(n) = \dim E_n = 2K'(n)$) and since $a(u) \in \widetilde{S}^0$, we get that $M(u,\cdot) \in \widetilde{S}^1$. Actually, decomposition (4.2.11) shows that $M(u,\cdot) \in \widetilde{S}^1_{\mathrm{sc}}$ since we may write it as the sum of $\begin{bmatrix} 0 & \omega_m(n)\mathrm{Id}_{K'(n)} \\ -(1+a(u))^2\omega_m(n)\mathrm{Id}_{K'(n)} & 0 \end{bmatrix}$, which is scalar by blocks, and of a contribution of order $-\infty$. Moreover the coefficients of $M(u,n)$ satisfy condition $C(\kappa,r)$.

According to definition 3.2.2, $\widetilde{\mathrm{Op}}(M(u,\cdot))u$ is nothing but the right hand side of (4.2.4). We may thus write this equation

$$\partial_t u = \widetilde{\mathrm{Op}}(M(u,\cdot))u. \tag{4.2.13}$$

Let us introduce the energy of solutions of (4.2.13) that we shall consider. We denote by $\widetilde{\Lambda}_m$ the operator $\widetilde{\Lambda}_m = \widetilde{\mathrm{Op}}(\omega_m(n)\mathrm{I}_{K(n)})$ acting on $L^2(\mathbb{S}^1, G)$, so that $\widetilde{\Lambda}_m \Pi_n = \omega_m(n)\Pi_n$. For s large enough we set

$$\Theta_0^s(u(t,\cdot)) = 2\langle \widetilde{\Lambda}_m^s \mathrm{Op}(Q_\chi(u;\cdot))u, \widetilde{\Lambda}_m^s \mathrm{Op}(Q_\chi(u;\cdot))u\rangle \tag{4.2.14}$$

where $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ close to 0, χ even, $\mathrm{Supp}\,\chi$ small enough, and where $Q_\chi \in \widetilde{\Sigma}_1^{0,\nu}$ (for some $\nu \in \mathbb{R}_+$) is defined from Q in corollary 3.3.3 (see also (3.2.5)). Because of (4.2.9)

$$v(Q_\chi) = 0, v'(Q_\chi) = \kappa. \tag{4.2.15}$$

The following lemma asserts that $\Theta_0^s(u)$ is indeed equivalent to $\|u\|_{H^s}^2$ for small u, and gives an alternative expression for $\Theta_0^s(u)$, which will be useful in the sequel.

Lemma 4.2.1 *There is $s_0 > 0$ and for any $s \geq s_0$ there are constants $C > 0, R_0 > 0$ such that for any $u \in H^s(\mathbb{S}^1, G)$ with $\|u\|_{H^{s_0}} < R_0$, one has*

$$C^{-1}\|u\|_{H^s}^2 \leq \Theta_0^s(u) \leq C\|u\|_{H^s}^2. \tag{4.2.16}$$

Moreover, we may find a self-adjoint scalar symbol $c(u,\cdot) \in \widetilde{\Sigma}_1^{2s,\nu}$, for some $\nu > 0$ independent of s, with $v(c) \geq \kappa$, and satisfying condition $C(\kappa, r)$, such that if $\tilde{u} = \mathrm{Op}(Q_\chi(u;\cdot))u$
(4.2.17)
$$\Theta_0^s(u) = \langle \widetilde{\Lambda}_m^s \mathrm{Op}((1+a_\chi)(u;\cdot))\tilde{u}, \widetilde{\Lambda}_m^s \tilde{u}\rangle + \langle \widetilde{\Lambda}_m^s \tilde{u}, \widetilde{\Lambda}_m^s \mathrm{Op}((1+a_\chi)(u;\cdot))\tilde{u}\rangle + \langle \mathrm{Op}(c(u;\cdot))\tilde{u}, \tilde{u}\rangle.$$

Proof: We prove first (4.2.17). Remark that the left hand side and the sum of the first two brackets in the right hand side of (4.2.17) are real, so if we find a symbol c satisfying (4.2.17), the equality remains true replacing c by $\frac{1}{2}(c + c^\bullet)$ where $c^\bullet$ is defined by (2.2.1). In other words, as soon as we have found a c, we can construct from it a self-adjoint one.

Compute the difference between $\frac{1}{2}\Theta_0^s(u)$ and the first bracket in the right hand side of (4.2.17). We get

$$-\langle \widetilde{\Lambda}_m^{2s} \mathrm{Op}(a_\chi(u;\cdot))\tilde{u}, \tilde{u}\rangle. \tag{4.2.18}$$

We may always write $\widetilde{\Lambda}_m^{2s}$ as a paradifferential operator associated to the symbol of $\Sigma_{0,0}^{2s,0}$ given by

$$\chi\Big(\frac{n_0 - n_1}{n_0 + n_1}\Big)\Big(\frac{\omega_m(n_0) + \omega_m(n_1)}{2}\Big)^{2s}.$$

Moreover a_χ defined from a in corollary 3.3.3 belongs to $\widetilde{\Sigma}_1^{0,\nu}$ for some $\nu \in \mathbb{R}_+$. By corollary 3.3.5 (i), we may thus write (4.2.18) as $\langle \mathrm{Op}(c(u;\cdot))\tilde{u}, \tilde{u}\rangle$ for some symbol $c \in \widetilde{\Sigma}_1^{2s,\nu}$, for a new value of ν independent of s. This gives (4.2.17).

Before starting the proof of (4.2.16), let us express u in function of $\tilde{u}$ and conversely. Denote

$$P_0(n) = P(0,n) = \begin{bmatrix} \mathrm{I}_{K'(n)} & \mathrm{I}_{K'(n)} \\ i\mathrm{I}_{K'(n)} & -i\mathrm{I}_{K'(n)} \end{bmatrix},\ Q_0(n) = Q(0,n) = \frac{i}{2}\begin{bmatrix} -i\mathrm{I}_{K'(n)} & -\mathrm{I}_{K'(n)} \\ -i\mathrm{I}_{K'(n)} & \mathrm{I}_{K'(n)} \end{bmatrix}. \tag{4.2.19}$$

If we denote $\sigma_0(u;n) = Q_\chi(u;n) - Q_{0,\chi}(n)$, we get a symbol in $\widetilde{\Sigma}_1^{0,\nu}$ for some ν, with $v(\sigma_0) \geq \kappa$, satisfying condition $C(\kappa, r)$, such that by definition of $\tilde{u}$

$$\tilde{u} = Q_0 u + \mathrm{Op}(\sigma_0(u;\cdot))u, \tag{4.2.20}$$

where for short we write Q_0 for $\widetilde{\mathrm{Op}}(Q_0(\cdot)) = \mathrm{Op}(Q_{0,\chi}(\cdot))$. Multiplying by $P_0 = Q_0^{-1}$ we get, using the same type of notation convention,

$$u = P_0\tilde{u} + \mathrm{Op}(\tilde{\sigma}_0(u;\cdot))u \tag{4.2.21}$$

for another symbol $\tilde{\sigma}_0$ with $\tilde{\sigma}_0 \in \widetilde{\Sigma}_1^{0,\nu}$, $v(\tilde{\sigma}_0) \geq \kappa$, $\tilde{\sigma}_0$ satisfying $C(\kappa, r)$. Using proposition 2.1.3, we obtain that there are $C > 0, s_0 > 0$ and for any $s \geq s_0$, there is $R_0 > 0$ small enough such that for any $u \in H^s$ with $\|u\|_{H^{s_0}} < R_0$,

$$C^{-1}\|\tilde{u}\|_{H^s} \leq \|u\|_{H^s} \leq C\|\tilde{u}\|_{H^s}, \tag{4.2.22}$$

since the last terms in (4.2.20), (4.2.21) are $O(\|u\|_{H^{s_0}}^\kappa \|u\|_{H^s}),\ u \to 0$. If we apply proposition 2.1.3 to the operators of order $2s$ $\widetilde{\Lambda}_m^{2s}\mathrm{Op}(a_\chi(u;\cdot))$ and $\mathrm{Op}(c(u;\cdot))$, we see that there is a new value of s_0, independent of the order of these operators, such that for $s \geq s_0$ there is $C_s > 0$ so that (4.2.18) as well as the last bracket in (4.2.17), are smaller than $C_s\|u\|_{H^{s_0}}^\kappa \|u\|_{H^s}^2$. This shows that

$$\Theta_0^s(u) - 2\langle \widetilde{\Lambda}_m^s \tilde{u}, \widetilde{\Lambda}_m^s \tilde{u}\rangle = O(\|u\|_{H^{s_0}}^\kappa \|u\|_{H^s}^2),\ u \to 0.$$

Inequalities (4.2.16) follow from that and (4.2.22). □

The interest of the preceding lemma is that it gives for Θ_0 an expression in terms of $\tilde{u}$, and the equation written on $\tilde{u}$ will be essentially diagonal. Let us introduce some more notations. We set

$$D(u,n) = Q(u,n)M(u,n)P(u,n) = i(1+a(u))^2 \begin{bmatrix} \lambda_m(n) & 0 \\ 0 & -\lambda_m(n) \end{bmatrix}. \tag{4.2.23}$$

We write also

$$D_0(n) = D(0,n),\ D^\kappa(u,n) = D(u,n) - D_0(n) \tag{4.2.24}$$

so that $D^\kappa(u,n) \in \widetilde{S}_{\mathrm{sc}}^1$ with valuation larger or equal to κ. Moreover $D(u,\cdot)$ satisfies condition $C(\kappa, r)$. We set also

$$M_0(n) = M(0,n),\ M^\kappa(u,n) = M(u,n) - M_0(n) \tag{4.2.25}$$

so that $M^\kappa(u,n)$ is an element of $\widetilde{S}_{\mathrm{sc}}^1$ of valuation larger or equal to κ. In the same way, the expressions

$$P^\kappa(u,n) = P(u,n) - P_0(n),\ Q^\kappa(u,n) = Q(u,n) - Q_0(n) \tag{4.2.26}$$

are symbols of $\widetilde{S}_{\mathrm{sc}}^0$, with valuations larger or equal to κ.

Lemma 4.2.2 *There is some $\nu \in \mathbb{R}_+$ and there are symbols $b_0(u;\cdot)$ in $\widetilde{\Sigma}_1^{0,\nu}$, $b_1(u;\cdot), \tilde{b}_1(u;\cdot)$ in $\widetilde{\Sigma}_1^{1,\nu}$ with $v(b_0), v(\tilde{b}_1) \geq \kappa$, there are operators $R, \tilde{R}$ in $\widetilde{\mathcal{R}}^{0,\nu}$, with $v(R), v(\tilde{R}) \geq \kappa + 1$, satisfying condition $C(\kappa, r)$, such that one may write for all $u \in H^s(\mathbb{S}^1, G)$ solution of (4.2.13)*

$$\frac{\partial u}{\partial t} = P_0 D_0 \tilde{u} + \mathrm{Op}(\tilde{b}_1(u;\cdot))u + \tilde{R}(u) \tag{4.2.27}$$

$$\frac{\partial \tilde{u}}{\partial t} = \mathrm{Op}(b_1(u;\cdot))u + R(u) \tag{4.2.28}$$

$$\mathrm{Op}((1 + a_\chi)(u;\cdot))\frac{\partial \tilde{u}}{\partial t} = \mathrm{Op}(D_\chi(u;\cdot))\tilde{u} + \mathrm{Op}(b_0(u;\cdot))u + R(u) \tag{4.2.29}$$

where we denoted by D_0 the operator $\widetilde{\mathrm{Op}}(D_0(n))$.

Proof: Let us show first (4.2.27). We apply corollary 3.3.3 to (4.2.13). We get

$$\frac{\partial u}{\partial t} = \mathrm{Op}(M_\chi(u;\cdot))u + \mathrm{Op}(\tilde{b}_0(u;\cdot))u + \tilde{R}(u) \tag{4.2.30}$$

where $\tilde{b}_0 \in \widetilde{\Sigma}_p^{0,\nu}$, $\tilde{R} \in \widetilde{\mathcal{R}}^{0,\nu}$ for some $\nu \in \mathbb{R}_+$, $v(\tilde{b}_0) \geq \kappa$, $v(\tilde{R}) \geq \kappa+1$, $\tilde{b}_0$ and $\tilde{R}$ satisfying condition $C(\kappa, r)$. Using (4.2.25), we further decompose $\mathrm{Op}(M_\chi(u;\cdot)) = M_0 + \mathrm{Op}(M_\chi^\kappa(u;\cdot))$, where M_0 denotes for short the operator with symbol $M_0(n)$. Since $M_\chi^\kappa(u,n) \in \widetilde{\Sigma}_1^{1,\nu}$, satisfies $v(M_\chi^\kappa) \geq \kappa$, and verifies condition $C(\kappa, r)$, we just have, to deduce (4.2.27) from (4.2.30), to express $M_0 u$ in terms of $\tilde{u}$. This follows from (4.2.21) together with the expression $M_0 P_0 = P_0 D_0$, which is a consequence of (4.2.23) and (4.2.8).

We shall prove now (4.2.28) and (4.2.29). We compute first

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial}{\partial t}[\mathrm{Op}(Q_\chi(u;\cdot))u] = \mathrm{Op}(Q_\chi(u;\cdot))\frac{\partial u}{\partial t} + \mathrm{Op}(Q'_\chi(U;\cdot))u \tag{4.2.31}$$

where $U = (u, \partial_t u)$ and $Q'(U,\cdot)$ is the symbol obtained by time derivation of $Q(u,\cdot)$. Let us show, using the equation, that $Q'_\chi(U;\cdot)$ is an element of $\widetilde{\Sigma}_1^{0,\nu}$ for some ν, satisfying $v(Q'_\chi(u;\cdot)) \geq \kappa$ and verifying condition $C(\kappa, r)$. By (4.2.7) we may write $Q'_\chi(U;\cdot)$ as a finite sum indexed by $p \geq \kappa$ of quantities of type

$$a_{p,\chi}(\partial_t u, u, \ldots, u; n_0, n_{p+1}) \begin{bmatrix} \frac{1}{2}\mathrm{I}_{K'(n_{p+1})} & 0 \\ \frac{1}{2}\mathrm{I}_{K'(n_{p+1})} & 0 \end{bmatrix}$$

where a_p is the component homogeneous of degree p in the expansion of a. If we plug in this expression (4.2.30), we see using corollary 3.3.6 (i) and (ii) that we get a contribution of type $\mathrm{Op}(b_0(u;\cdot))u + R(u)$, like the last two terms in the right hand side of (4.2.29). In particular, such terms are of the form of the right hand side of (4.2.28). To finish the proof of (4.2.28), we just have to study the first term in the right hand side of (4.2.31). If we replace in that term $\partial_t u$ by (4.2.30) and use corollaries 3.3.5 (i) and 3.3.6 (iii), we obtain that this contribution is again of the same form as the right hand side of (4.2.28). Let us prove (4.2.29), making act

$\mathrm{Op}((1+a_\chi)(u;\cdot))$ on (4.2.31). We have seen already that the last term in the right hand side of (4.2.31) has the structure of the last two terms in the right hand side of (4.2.29). This remains true if we make act $\mathrm{Op}((1+a_\chi)(u;\cdot))$ on it, by corollary 3.3.5 (i) and corollary 3.3.6 (iii). So, we just have to study, using (4.2.30)

$$
\begin{aligned}
\mathrm{Op}((1+a_\chi)(u;\cdot))\mathrm{Op}(Q_\chi(u;\cdot))\frac{\partial u}{\partial t} &= \mathrm{Op}((1+a_\chi)(u;\cdot))\mathrm{Op}(Q_\chi(u;\cdot))\mathrm{Op}(M_\chi(u;\cdot))u \\
&\quad + \mathrm{Op}((1+a_\chi)(u;\cdot))\mathrm{Op}(Q_\chi(u;\cdot))\mathrm{Op}(\tilde{b}_0(u;\cdot))u \\
&\quad + \mathrm{Op}((1+a_\chi)(u;\cdot))\mathrm{Op}(Q_\chi(u;\cdot))\tilde{R}(u).
\end{aligned} \tag{4.2.32}
$$

Again by corollaries 3.3.5 (i) and 3.3.6 (iii), the last two terms give a contribution to the last two terms in (4.2.29). Since a is a scalar symbol we may, by corollary 3.3.5 (ii), commute in the first term in the right hand side of (4.2.32), $\mathrm{Op}((1+a_\chi)(u;\cdot))$ and $\mathrm{Op}(Q_\chi(u;\cdot))\mathrm{Op}(M_\chi(u;\cdot))$, up to errors that may be incorporated inside the $\mathrm{Op}(b_0(u;\cdot))u$ term in (4.2.29). We are thus reduced to

$$
\mathrm{Op}(Q_\chi(u;\cdot))\mathrm{Op}(M_\chi(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))u. \tag{4.2.33}
$$

We apply corollary 3.3.4 to the symbols P and Q satisfying (4.2.8). Using also corollary 3.3.5 (i) and corollary 3.3.6 (iii), we obtain that (4.2.33) may be written as

$$
[\mathrm{Op}(Q_\chi(u;\cdot))\mathrm{Op}(M_\chi(u;\cdot))\mathrm{Op}(P_\chi(u;\cdot))]\mathrm{Op}(Q_\chi(u;\cdot))u
$$

up again to contributions to the last two terms in (4.2.29). To conclude the proof, we just have to apply again corollary 3.3.4 to the bracket in the above formula, making use of the first equality (4.2.23) and of corollaries 3.3.5 (i), 3.3.6 (iii) and 3.3.7 (i). □

We want to obtain a formula giving the time derivative of expressions generalizing the first term in the right hand side of (4.2.17). We introduce first some notations. We shall consider symbols $c \in \Sigma^{d,\nu}_{p,N_0}$ satisfying the following conditions

$$
c(U;\cdot) = c'(U;\cdot) + c''(U;\cdot) \text{ with } c'' \in \Sigma^{d-1,\nu}_{p,N_0} \text{ and self-adjoint,} \tag{4.2.34}
$$

$$
\begin{aligned}
&c'(U;\cdot) \text{ is self-adjoint and for any } n_0, n_{p+1} \geq \tau+1, \\
&c'(U;n_0,n_{p+1}) = \begin{bmatrix} c_{11}(U;n_0,n_{p+1}) & 0 \\ 0 & c_{22}(U;n_0,n_{p+1}) \end{bmatrix} \text{ with } 2\times 2 \text{ matrices } c_{11}, c_{22}.
\end{aligned} \tag{4.2.35}
$$

(Remind that our symbols of $\Sigma^{d,\nu}_{p,N_0}$ are 4×4 matrices when evaluated at (n_0, n_{p+1}) with $n_0, n_{p+1} \geq \tau+1$).

When $c \in \Sigma^{d,\nu}_{p,N_0}$ we shall associate to it the following function

$$
c_{M_0}(u;n_0,n_{p+1}) = \sum_{j=1}^{p} c(u,\dots,M_0 u,\dots,u;n_0,n_{p+1}) \tag{4.2.36}
$$

where as before M_0 denotes the operator with symbol $M_0(n)$, and where the term $M_0 u$ is the jth argument of the general term of the sum. We first prove a lemma.

Lemma 4.2.3 *Let $\nu \in \mathbb{R}_+$. There is $\nu' \in \mathbb{R}_+$ such that for any $d \in \mathbb{R}, N_0 \in \mathbb{N}^*, p \in \mathbb{N}, c \in \Sigma^{d,\nu}_{p,N_0}$, one can find a symbol $e_1 \in \widetilde{\Sigma}^{d,\nu'}_{N_0}$ with $v(e_1) \geq \kappa + p$ and $R_1 \in \widetilde{\mathcal{R}}^{d,\nu'}$ with $v(R_1) \geq \kappa + p + 1$, such that for any smooth enough solution u of (4.2.13), any smooth enough v*

$$\mathrm{Op}(\frac{\partial}{\partial t} c(u, \dots, u; \cdot))v = \mathrm{Op}(c_{M_0}(u; \cdot))v + \mathrm{Op}(e_1(u; \cdot))v + R_1(u, v). \tag{4.2.37}$$

Moreover, if p is odd and $p \geq \kappa$, then e_1 satisfies condition $C(\kappa, r)$.

Proof: The left hand side of (4.2.37) is a sum of expressions

$$\mathrm{Op}(c(u, \dots, \frac{\partial u}{\partial t}, \dots, u; \cdot))v. \tag{4.2.38}$$

We use for $\frac{\partial u}{\partial t}$ expression (4.2.30) and decomposition (4.2.25). We get

$$\frac{\partial u}{\partial t} = M_0 u + \mathrm{Op}(M^\kappa_\chi(u; \cdot))u + \mathrm{Op}(\tilde{b}_0(u; \cdot))u + \tilde{R}(u).$$

When we plug this decomposition inside (4.2.38), we get from the $M_0 u$ term, according to (4.2.36), a contribution to the first term in the right hands side of (4.2.37). The remaining terms in the above expression of $\frac{\partial u}{\partial t}$ contribute to the last two terms in (4.2.37), using (i) and (ii) of corollary 3.3.6. $\square$

Let us state now the main proposition.

Proposition 4.2.4 *Let $\nu \in \mathbb{R}_+, p \in \mathbb{N}, N_0 \in \mathbb{N}^*$ be given. There is $\nu' \in \mathbb{R}_+$ and for any $d \in \mathbb{R}$, for any symbol $c \in \Sigma^{d,\nu}_{p,N_0}$ satisfying (4.2.34), (4.2.35), one can find*

- *a self-adjoint symbol $e \in \widetilde{\Sigma}^{d,\nu'}_{N_0}$ with $v(e) \geq p + \kappa$,*
- *an operator $R \in \widetilde{\mathcal{R}}^{d,\nu'}$ satisfying $v(R) \geq p + \kappa + 1$,*

such that for any smooth enough u satisfying equation (4.2.13) one has, denoting $c(u; \cdot) = c(u, \dots, u; \cdot)$,

(4.2.39)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle[\mathrm{Op}(c(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot)) + \mathrm{Op}((1+a_\chi)(u;\cdot))^*\mathrm{Op}(c(u;\cdot))]\tilde{u}, \tilde{u}\rangle \\ = \langle \mathrm{Op}(c_{M_0}(u;\cdot))\tilde{u}, \tilde{u}\rangle + \langle[\mathrm{Op}(c(u;\cdot))D_0 + D_0^*\mathrm{Op}(c(u;\cdot))]\tilde{u}, \tilde{u}\rangle \\ + \langle \mathrm{Op}(e(u;\cdot))\tilde{u}, \tilde{u}\rangle + (\langle R(u), u\rangle + \langle u, R(u)\rangle).\end{aligned}$$

Moreover, if p is odd, $p \geq \kappa$ then c_{M_0}, e, R satisfy condition $C(\kappa, r)$.

Proof: Remark that since c is self-adjoint, so is c_{M_0} defined by (4.2.36). So the left hand side and the first two terms in the right hand side of (4.2.39) are real valued. Consequently, it is enough to prove (4.2.39) for some non necessarily self-adjoint symbol e, and replacing $(\langle R(u), u\rangle + \langle u, R(u)\rangle)$

by $(\langle R_1(u),u\rangle+\langle u,R_2(u)\rangle)$ for some R_1,R_2 satisfying the same conditions as R. Then taking real parts, we replace e by $\frac{e+e^\bullet}{2}$ and R_j by $\frac{R_1+R_2}{2}$ to get (4.2.39).

Let us show that we can write as the right hand side of (4.2.39) the time derivative

$$\begin{aligned}&\frac{d}{dt}\langle \mathrm{Op}(c(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))\tilde{u},\tilde{u}\rangle =\\ &\langle \mathrm{Op}(\frac{d}{dt}c(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))\tilde{u},\tilde{u}\rangle+\langle \mathrm{Op}(c(u;\cdot))\mathrm{Op}(\frac{d}{dt}a_\chi(u;\cdot))\tilde{u},\tilde{u}\rangle\\ &+\langle \mathrm{Op}(c(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))\frac{d}{dt}\tilde{u},\tilde{u}\rangle+\langle \mathrm{Op}(c(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))\tilde{u},\frac{d}{dt}\tilde{u}\rangle.\end{aligned}\tag{4.2.40}$$

The idea of the proof is the following: we shall express $\frac{\partial\tilde{u}}{\partial t}$ using (4.2.28) or (4.2.29). The linear contributions coming from these expressions will give the first two terms in the right hand side of (4.2.39). The contributions which are at least of order κ in u will contribute to the last two terms. The key point will be not to loose derivatives, i.e. to check that e is of order d and not $d+1$. This will follow from the fact that $\mathrm{Op}(e(u;\cdot))$ will be expressed from commutators of operators with commuting symbols. Symbolic calculus will thus bring the needed gain of one derivative. Let us proceed with the implementation of such a strategy.

Study of first term in RHS of (4.2.40)

Let us consider

$$\mathrm{Op}(\frac{d}{dt}c(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))\tilde{u}.$$

By lemma 4.2.3, we may write this as

$$\mathrm{Op}(c_{M_0}(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))\tilde{u}+\mathrm{Op}(e_1(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))\tilde{u}+R_1(u,\mathrm{Op}((1+a_\chi)(u;\cdot))\tilde{u}).\tag{4.2.41}$$

The first term gives on one hand the first term in the right hand side of (4.2.39), and on the other hand a contribution $\mathrm{Op}(c_{M_0}(u;\cdot))\mathrm{Op}(a_\chi(u;\cdot))\tilde{u}$. Using corollary 3.3.5 (i), we see that this expression can be incorporated in the $\mathrm{Op}(e(u;\cdot))\tilde{u}$ term in (4.2.39). Remark that the index ν' given by corollary 3.3.5 is independent of the order d of c. The second term in (4.2.41) gives similarly a contribution to the e-term in (4.2.39). In the last term, we express $\tilde{u}$ from u using (4.2.20). From corollaries 3.3.5 (i) and 3.3.7 (i), we see that we obtain a contribution $\langle R(u),\tilde{u}\rangle$ for some R satisfying the requirements of the statement of proposition 4.2.4. If we express $\tilde{u}$ from u by (4.2.20) and use (iii) of corollary 3.3.6, we see that we obtain a contribution to the fourth term in the right hand side of (4.2.39).

Study of second term in RHS of (4.2.40)

If we apply lemma 4.2.3 to the symbol of order 0 a_χ, we see that

$$\mathrm{Op}(\frac{d}{dt}a_\chi(u;\cdot))\tilde{u}=\mathrm{Op}(a_{\chi,M_0}(u;\cdot))\tilde{u}+\mathrm{Op}(e_1(u;\cdot))\tilde{u}+R_1(u,\tilde{u})$$

where $a_{\chi,M_0}\in\widetilde{\Sigma}_1^{0,\nu}$, $e_1\in\widetilde{\Sigma}_1^{0,\nu'}$, $R_1\in\widetilde{\mathcal{R}}^{0,\nu'}$ for some $\nu'\in\mathbb{R}_+$, with moreover $v(a_{\chi,M_0})\geq\kappa$, $v(e_1)\geq 2\kappa$, $v(R_1)\geq 2\kappa+1$. If we make act on the left $\mathrm{Op}(c(u;\cdot))$ and use as before corollaries 3.3.5 (i), 3.3.6 (iii), (4.2.20) and corollary 3.3.7 (i), we obtain a contribution to the third and fourth terms in (4.2.39).

Study of third and fourth terms in RHS of (4.2.40)

We write the sum of the last two terms in (4.2.40) as

$$2\mathrm{Re}\,\langle \mathrm{Op}(c(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))\frac{d\tilde{u}}{dt},\tilde{u}\rangle + \langle [\mathrm{Op}(c(u;\cdot)),\mathrm{Op}(a_\chi(u;\cdot))]\tilde{u},\frac{d\tilde{u}}{dt}\rangle \tag{4.2.42}$$

using that c and a_χ are self-adjoint symbols. We may apply corollary 3.3.5 (ii) to the bracket in (4.2.42), since a_χ is scalar and so commutes to c. There is ν', independent of d, and a symbol $b \in \widetilde{\Sigma}^{d-1,\nu'}_{N_0}$ with $v(b) \geq \kappa + p$ such that the last term in (4.2.42) equals

$$\langle \mathrm{Op}(b(u;\cdot))\tilde{u},\frac{d\tilde{u}}{dt}\rangle.$$

Using (4.2.28), we reduce ourselves to the study of

$$\langle \tilde{u},\mathrm{Op}(b(u;\cdot))^*(\mathrm{Op}(b_1(u;\cdot))u + R(u))\rangle. \tag{4.2.43}$$

Using, as in the study of the preceding cases, (4.2.20), and corollaries 3.3.5 (i) and 3.3.6 (iii), we may write this expression as a contribution to the third and fourth terms in the right hand side of (4.2.39), using that the sum of the orders of the involved symbols is at most d.

Let us study now the first term in (4.2.42). We write using (4.2.29)

$$\begin{aligned}\mathrm{Op}(c(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot))\frac{d\tilde{u}}{dt} &= \mathrm{Op}(c(u;\cdot))\mathrm{Op}(D_\chi(u;\cdot))\tilde{u}\\ &\quad+\mathrm{Op}(c(u;\cdot))\mathrm{Op}(b_0(u;\cdot))u\\ &\quad+\mathrm{Op}(c(u;\cdot))R(u).\end{aligned} \tag{4.2.44}$$

The contribution of the last two terms to the first duality bracket in (4.2.42) is of form the conjugate of (4.2.43), since the sum of the orders of the symbols is at most d, and $R(\cdot) \in \widetilde{\mathcal{R}}^{0,\nu}$, and has been already treated. To study the first term in the right hand side of (4.2.44), where D_χ is a symbol of order 1, remind decomposition (4.2.24), which allows us to write

$$D_\chi(u;\cdot) = D_{0,\chi}(\cdot) + D^\kappa_\chi(u;\cdot). \tag{4.2.45}$$

We study first the contribution of the last term i.e.

$$\begin{aligned}&2\mathrm{Re}\,\langle \mathrm{Op}(c(u;\cdot))\mathrm{Op}(D^\kappa_\chi(u;\cdot))\tilde{u},\tilde{u}\rangle =\\ &\quad\langle [\mathrm{Op}(c(u;\cdot))\mathrm{Op}(D^\kappa_\chi(u;\cdot)) + \mathrm{Op}(D^\kappa_\chi(u;\cdot))^*\mathrm{Op}(c(u;\cdot))^*]\tilde{u},\tilde{u}\rangle.\end{aligned} \tag{4.2.46}$$

Remind decomposition (4.2.34) of c. Since $c'' \in \Sigma^{d-1,\nu}_{p,N_0}$, we may write by corollary 3.3.5 (i) $\mathrm{Op}(c''(u;\cdot))\mathrm{Op}(D^\kappa_\chi(u;\cdot)) = \mathrm{Op}(g(u;\cdot))$ for a new symbol $g \in \widetilde{\Sigma}^{d,\nu'}_{N_0}$ with ν' independent of d and $v(g) \geq p+\kappa$. This term will give in (4.2.46) a contribution which can be treated as (4.2.43). The c' contribution to (4.2.46) may be written, since c' is self-adjoint

$$\langle [\mathrm{Op}(c'(u;\cdot))\mathrm{Op}(D^\kappa_\chi(u;\cdot)) + \mathrm{Op}(D^\kappa_\chi(u;\cdot))^*\mathrm{Op}(c'(u;\cdot))]\tilde{u},\tilde{u}\rangle. \tag{4.2.47}$$

By (4.2.23), (4.2.24) and (4.2.11), we may decompose

$$D^\kappa(u,n) = D^{\kappa\prime}(u,n) + \hat{D}^\kappa(u,n)$$

with

$$D^{\kappa\prime}(u,n) = i(2a(u)+a(u)^2)\omega_m(n)\begin{bmatrix}\mathrm{I}_{K'(n)} & 0\\ 0 & -\mathrm{I}_{K'(n)}\end{bmatrix}$$
$$\hat{D}^{\kappa}(u,n) \text{ of order } -\infty.$$

The contribution of $\hat{D}^{\kappa}(u,n)$ to (4.2.47) may be treated as expression (4.2.43). Since we may write $D^{\kappa\prime}(u,n)^* = -D^{\kappa\prime}(u,n)$, (ii) of proposition 3.2.7 shows that $\mathrm{Op}(D_\chi^{\kappa\prime}(u;\cdot))^* = -\mathrm{Op}(D_\chi^{\kappa\prime}(u;\cdot))$ modulo an operator of order zero, whose contribution may be treated as (4.2.43). Consequently, we are left with

$$\langle[\mathrm{Op}(c'(u;\cdot)),\mathrm{Op}(D^{\kappa\prime}(u;\cdot))]\tilde{u},\tilde{u}\rangle. \tag{4.2.48}$$

Remark now that by assumption (4.2.35) and the expression of $D^{\kappa\prime}$, we have $c'(u,\cdot)\circ D^{\kappa\prime}(u,\cdot) = D^{\kappa\prime}(u,\cdot)\circ c'(u,\cdot)$ (for large enough phase arguments of the symbols). We may therefore apply corollary 3.3.5 (ii) to write the commutator as an operator associated to a symbol in $\widetilde{\Sigma}^{d,\nu'}_{N_0}$, of valuation larger or equal to $\kappa+p$, for some ν' independent of d. Reasoning as for (4.2.43), we get again a contribution to the last two terms in (4.2.39).

To finish the proof, we just have to remark that the contribution to the first term in (4.2.42) obtained plugging the first term in the right hand side of (4.2.45) inside the first term in the right hand side of (4.2.44) gives the second term in the right hand side of (4.2.39). This concludes the proof of the proposition. □

Proposition 4.2.5 *Let $\nu\in\mathbb{R}_+$. There is $\nu'\in\mathbb{R}_+$ and for any $p\in\mathbb{N}^*$, $d\in\mathbb{R}$, $\widetilde{M}\in\mathcal{R}^{d,\nu}_p$, there are operators $R_1\in\widetilde{\mathcal{R}}^{d+1,\nu'}, R_2\in\widetilde{\mathcal{R}}^{0,\nu'}$ with $v(R_1)\geq\kappa+p, v(R_2)\geq\kappa+1$, such that for any smooth enough u solving equation (4.2.13)*

$$\begin{aligned}\frac{d}{dt}\langle\widetilde{M}(u,\dots,u),u\rangle = \sum_{j=1}^{p}\langle\widetilde{M}(u,\dots,M_0u,\dots,u),u\rangle + \langle M_0^*\widetilde{M}(u,\dots,u),u\rangle\\ + \langle R_1(u),u\rangle + \langle\widetilde{M}(u,\dots,u),R_2(u)\rangle.\end{aligned} \tag{4.2.49}$$

Proof: We compute first $\widetilde{M}(\frac{du}{dt},u,\dots,u)$ using formulas (4.2.30) and decomposing

$$\mathrm{Op}(M_\chi(u;\cdot)(u;\cdot)) = M_0u + \mathrm{Op}(M_\chi^\kappa(u;\cdot)(u;\cdot)).$$

Using corollary 3.3.7, we get a contribution to the first and third terms in the right hand side of (4.2.49). In the same way, we get from $\langle\widetilde{M}(u,\dots,u),\frac{du}{dt}\rangle$, using corollary 3.3.6 (iii) contributions to the last two terms in (4.2.49). □

4.3 Refined energy inequalities and proof of the main theorem

The objective of this subsection is to prove proposition 4.3.2 below, which will imply theorem 1.1.1. Remind that we defined in (4.2.14) for a solution u of equation (4.2.4) the quantity $\Theta_0^s(u(t,\cdot))$ which, as long as $\|u(t,\cdot)\|_{H^s}$ remains small enough, is equivalent to $\|u(t,\cdot)\|_{H^s}^2$. We shall see that $\frac{d}{dt}\Theta_0^s(u(t,\cdot))$ may be written essentially as $\langle \mathrm{Op}(a(u;\cdot))\tilde{u},\tilde{u}\rangle$ for a symbol a of order $2s$ and valuation κ. We shall next find a correction $\Theta_1^s(u(t,\cdot))$ so that $\frac{d}{dt}(\Theta_0^s(u(t,\cdot))-\Theta_1^s(u(t,\cdot)))$ may be written as $\langle \mathrm{Op}(b(u;\cdot))\tilde{u},\tilde{u}\rangle$ with b of order $2s$ and valuation $r-1>\kappa$. This gain on the valuation will give us the long time existence result we look for. The correction Θ_1^s will be constructed solving an equation on symbols involving the right hand side of (4.2.39). This is the main technical part of this subsection.

Let us first recall some notations, and a result of [10] that will play a crucial role. Remind from subsection 1.2 that the large eigenvalues of $P=\sqrt{-\Delta+V}$ come by pairs $\omega_-(n)\leq\omega_+(n)$ having the same asymptotics (1.2.1). We denote as before by $\omega(\cdot)$ a symbol on $\mathbb{R}_+$ with asymptotics (1.2.1) at infinity. We fix a large enough integer τ so that the spectrum $\mathcal{H}$ of P may be written

$$\mathcal{H}=(\mathcal{H}\cap I_\tau)\cup\bigcup_{n=\tau+1}^{+\infty}(\mathcal{H}\cap I_n), \tag{4.3.1}$$

where for $n\geq\tau+1$, I_n are disjoint intervals of length $O(n^{-\infty})$ centered at $\omega(n)$ and containing $\omega_-(n)$ and $\omega_+(n)$, and where I_τ contains the small eigenvalues. We set $\widetilde{\mathcal{H}}=\mathcal{H}\cup\{\omega(n);n\in\mathbb{N}\}$, and write for $\widetilde{\mathcal{H}}$ a decomposition of form (4.3.1). The decomposition of $L^2(\mathbb{S}^1,\mathbb{R}^2)$ associated to (4.3.1) is given by (4.2.5). Let us recall a special case of proposition 2.2.1 of [10]. We use notation (2.1.5).

Proposition 4.3.1 *For any $\xi\in\mathcal{H}$ (or $\widetilde{\mathcal{H}}$), denote by $n(\xi)$ the unique $n\in\mathbb{N}_\tau$ such that $\xi\in I_{n(\xi)}$. Let p be an odd positive integer. There is a zero measure subset $\mathcal{N}$ of $]0,+\infty[$ such that for any $m\in]0,+\infty[-\mathcal{N}$, there are $c>0$, $N_0\in\mathbb{N}$, so that for any $\xi_0,\dots,\xi_{p+1}\in\mathcal{H}$ (or $\widetilde{\mathcal{H}}$), any $0\leq q\leq p+1$*

$$\Big|\sum_{j=0}^{q}\sqrt{m^2+\xi_j^2}-\sum_{j=q+1}^{p+1}\sqrt{m^2+\xi_j^2}\Big|\geq c\mu(n(\xi_0),\dots,n(\xi_{p+1}))^{-N_0}. \tag{4.3.2}$$

From now on, we fix a value of m outside $\mathcal{N}$, and so an integer N_0. We shall state and prove a proposition relying on division by quantities of form (4.3.2). We need first to introduce some notations. If a is a paradifferential symbol, $a\in\Sigma^{d,\nu}_{p,N_0}$, remind that for any $u_1,\dots,u_p\in\mathcal{E}$, $n_0,n_{p+1}\in\mathbb{N}_\tau$, $a(u_1,\dots,u_p;n_0,n_{p+1})$ is a $K(n_0)\times K(n_{p+1})$ matrix, where for $n\in\mathbb{N}_\tau$, $K(n)$ is an even integer (and $K(n)=4$ if $n\geq\tau+1$). We can write a block decomposition of a involving $K(n_0)/2$ lines and $K(n_{p+1})/2$ columns

$$\begin{bmatrix} * & * \\ * & * \end{bmatrix}. \tag{4.3.3}$$

We shall consider the following two assumptions

(H_D) In (4.3.3) each block outside the diagonal is zero.

(H_{ND}) In (4.3.3) each block on the diagonal is zero.

In accordance with notations (4.2.36), if c^1 is a symbol in $\Sigma^{d,\nu}_{p,N_0}$ we shall set

$$c^1_{M_0}(u_1,\dots,u_p;n_0,n_{p+1}) = \sum_{j=1}^{p} c^1(u_1,\dots,M_0u_j,\dots,u_p;n_0,n_{p+1}). \tag{4.3.4}$$

Proposition 4.3.2 *Let $\nu\in\mathbb{R}_+$. There is $\nu'\in\mathbb{R}_+$ such that for any $d\in\mathbb{R}, p\in\mathbb{N}$ odd, $a\in\Sigma^{d,\nu}_{p,N_0}$ satisfying assumption (H_D) (resp. assumption (H_{ND})), we may find a symbol $c^1\in\Sigma^{d,\nu'}_{p,N_0}$ satisfying (H_D) (resp. a symbol $c^1\in\Sigma^{d-1,\nu'}_{p,N_0}$ satisfying (H_{ND})) such that*

(4.3.5)
$$\begin{aligned}c^1_{M_0}(u_1,\dots,u_p;n_0,n_{p+1}) + c^1(u_1,\dots,u_p;n_0,n_{p+1})D_0(n_{p+1}) - D_0(n_0)c^1(u_1,\dots,u_p;n_0,n_{p+1})\\ = a(u_1,\dots,u_p;n_0,n_{p+1}).\end{aligned}$$

Moreover, if a is self-adjoint, we may assume that c^1 is also self-adjoint.

Remark that the last statement follows from (4.3.5) and the fact that if c^1 satisfies (4.3.5), then $c^{1\bullet}$ defined by (2.2.1) satisfies also (4.3.5) with right hand side replaced by $a^\bullet$ (since $D(n)^* = -D(n)$).

The proof of (4.3.5) will use several lemmas. We remark first that we may extend c^1 and a, which are $\mathbb{R}$-mulitilinear maps in $(u_1,\dots,u_p)$ as $\mathbb{C}$-mulitilinear maps. This allows us to make the change of function $u_j\to P_0u_j$ in (4.3.5), where P_0 is defined in (4.2.19) and satisfies by (4.2.23) $P_0D_0 = M_0P_0$. This equation is thus equivalent to

(4.3.6)
$$\begin{aligned}\tilde c^1_{D_0}(u_1,\dots,u_p;n_0,n_{p+1}) + \tilde c^1(u_1,\dots,u_p;n_0,n_{p+1})D_0(n_{p+1}) - D_0(n_0)\tilde c^1(u_1,\dots,u_p;n_0,n_{p+1})\\ = \tilde a(u_1,\dots,u_p;n_0,n_{p+1})\end{aligned}$$

where we denoted

$$\begin{aligned}\tilde a(u_1,\dots,u_p;n_0,n_{p+1}) &= a(P_0u_1,\dots,P_0u_p;n_0,n_{p+1})\\ \tilde c^1(u_1,\dots,u_p;n_0,n_{p+1}) &= c^1(P_0u_1,\dots,P_0u_p;n_0,n_{p+1})\\ \tilde c^1_{D_0}(u_1,\dots,u_p;n_0,n_{p+1}) &= \sum_{j=1}^{p} c^1(P_0u_1,\dots,P_0D_0u_j,\dots,P_0u_p;n_0,n_{p+1}).\end{aligned} \tag{4.3.7}$$

We shall denote by $\Sigma^{d,\nu}_{p,N_0}(N)$ the space of functions a of type (2.1.10), defined on $(\mathcal{E}\otimes\mathbb{C})\times\cdots\times(\mathcal{E}\otimes\mathbb{C})\times\mathbb{N}_\tau\times\mathbb{N}_\tau$ instead of $\mathcal{E}\times\cdots\times\mathcal{E}\times\mathbb{N}_\tau\times\mathbb{N}_\tau$, which are $\mathbb{C}$-p-linear in $(u_1,\dots,u_p)$ and satisfy condition $(\mathrm{i})_\delta$ of definition 2.1.1 for some $\delta\in]0,1[$ small enough, and inequalities (2.1.12) only when $\alpha+\beta+\gamma\le N$. We endow this space with the norm $|a|^{d,\nu}_{p,N_0,N}$ given by the best constant in inequality (2.1.12). Of course, $\Sigma^{d,\nu}_{p,N_0}$ is the restriction of $\bigcap_N \Sigma^{d,\nu}_{p,N_0}(N)$ to real arguments $(u_1,\dots,u_p)$. If $c^1\in\Sigma^{d,\nu}_{p,N_0}(N)$ we denote by $L(c^1)$ the symbol defined by the left hand side of (4.3.6). Remind that by (4.2.23), (4.2.24), (4.2.11), the matrix $D_0(n)=D(0,n)$ may be decomposed as

$$D_0(n)=D_0'(n)+\hat D_0(n),\ D_0'(n)=i\omega_m(n)\begin{bmatrix}\mathrm{I}_{K'(n)} & 0\\ 0 & -\mathrm{I}_{K'(n)}\end{bmatrix}\tag{4.3.8}$$

where $\hat D_0(n)$ is a symbol of order $-\infty$. When $n=\tau$, we may take $D_0'(\tau)=0$. We then decompose

$$L(c^1)=L_0(c^1)+L_1(c^1)\tag{4.3.9}$$

with, if $U'=(u_1,\dots,u_p)$,
(4.3.10)
$$L_0(c^1)(U';n_0,n_{p+1})=\tilde c^1_{D_0}(U';n_0,n_{p+1})+\tilde c^1(U';n_0,n_{p+1})D_0'(n_{p+1})-D_0'(n_0)\tilde c^1(U';n_0,n_{p+1})$$

and

$$L_1(c^1)(U';n_0,n_{p+1})=\tilde c^1(U';n_0,n_{p+1})\hat D_0(n_{p+1})-\hat D_0(n_0)\tilde c^1(U';n_0,n_{p+1}).\tag{4.3.11}$$

Remark that L_1 sends $\Sigma^{d,\nu}_{p,N_0}(N)$ into $\Sigma^{-\infty,0}_{p,N_0}(N)$ since $\hat D_0$ is of order $-\infty$. On the other hand, if c^1 satisfies condition (H_D), $\tilde c^1(U';n_0,n_{p+1})$ commutes when $n_0,n_{p+1}\in\mathbb{N}_{\tau+1}$ to $D_0'(n_0)$ whence

$$L_0(c^1)(U';n_0,n_{p+1})=\tilde c^1_{D_0}(U';n_0,n_{p+1})+\tilde c^1(U';n_0,n_{p+1})(D_0'(n_{p+1})-D_0'(n_0)).\tag{4.3.12}$$

Remark that because of definition (4.2.10) of ω_m, $\omega_m(n_{p+1})-\omega_m(n_0)$ satisfies when $|n_{p+1}-n_0|\le\frac14(n_{p+1}+n_0)$ inequalities (3.3.15). This shows that if

$$\begin{aligned}\Sigma'^{d,\nu}_{p,N_0}&=\{a\in\bigcap_N\Sigma^{d,\nu}_{p,N_0}(N);a\text{ satisfies }(H_D)\}\\ \Sigma'^{d,\nu}_{p,N_0}(N)&=\Sigma'^{d,\nu}_{p,N_0}\cap\Sigma^{d,\nu}_{p,N_0}(N),\end{aligned}\tag{4.3.13}$$

then L_0 sends $\Sigma'^{d,\nu}_{p,N_0}(N)$ into $\Sigma'^{d,\nu+1}_{p,N_0}(N-1)$.

If c^1 satisfies assumption (H_{ND}), then for $n_0,n_{p+1}\in\mathbb{N}_{\tau+1}$,

$$\tilde c^1(U';n_0,n_{p+1})D_0'(n_0)=-D_0'(n_0)\tilde c^1(U';n_0,n_{p+1})$$

whence

$$L_0(c^1)(U';n_0,n_{p+1})=\tilde c^1_{D_0}(U';n_0,n_{p+1})+\tilde c^1(U';n_0,n_{p+1})(D_0'(n_{p+1})+D_0'(n_0)).\tag{4.3.14}$$

If we define

$$\begin{aligned}\Sigma''^{d,\nu}_{p,N_0}&=\{a\in\bigcap_N\Sigma^{d,\nu}_{p,N_0}(N);a\text{ satisfies }(H_{ND})\}\\ \Sigma''^{d,\nu}_{p,N_0}(N)&=\Sigma''^{d,\nu}_{p,N_0}\cap\Sigma^{d,\nu}_{p,N_0}(N),\end{aligned}\tag{4.3.15}$$

we obtain that L_0 sends $\Sigma''^{d,\nu}_{p,N_0}(N)$ in $\Sigma^{d+1,\nu}_{p,N_0}(N)$. Let us prove the following lemma:

Lemma 4.3.3 *(i) For any $d \in \mathbb{R}, \nu \in \mathbb{R}_+, p \in \mathbb{N}, N \in \mathbb{N}$, the operator L is injective on $\Sigma^{d,\nu}_{p,N_0}(N)$.*

(ii) Let F be a subspace of $\Sigma^{d,\nu}_{p,N_0}(N)$ such that there is a finite subset K of $\mathbb{N}_\tau \times \mathbb{N}_\tau$ so that for any $a \in F$, $a(\cdot; n_0, n_{p+1}) \equiv 0$ if $(n_0, n_{p+1}) \notin K$. Then F is stable by L and $L : F \to F$ is bijective.

Proof: (i) We denote by Π_n the spectral projector on the space $E_n \otimes \mathbb{C}$, where E_n is defined by the decomposition (4.2.5) of $L^2(\mathbb{S}^1;\mathbb{R}^2)$. We shall use the notation Π'_n for the similar projector acting on $L^2(\mathbb{S}^1;\mathbb{C})$. For every n, we denote by $(\omega(n,\ell))_\ell$ the $K'(n)$ eigenvalues of the restriction of $P = \sqrt{-\Delta + V}$ to the range of Π'_n acting on $L^2(\mathbb{S}^1;\mathbb{C})$. We choose an orthonormal basis of that range made of eigenfunctions of P associated to these eigenvalues (this is not in general a nice basis). We write

$$\Pi'_n = \sum_\ell \Pi'^{\ell}_n \tag{4.3.16}$$

the corresponding decomposition of Π'_n. The sum in (4.3.16) is finite, and for $n \geq \tau + 1$ made of only two terms as the range of Π'_n is two dimensional. We set $\omega_m(n,\ell) = \sqrt{m^2 + \omega(n,\ell)^2}$ and we have

$$\Lambda_m \Pi'^{\ell}_n = \omega_m(n,\ell)\Pi'^{\ell}_n \tag{4.3.17}$$

and $(\omega_m(n,\ell))_\ell$ are the eigenvalues of the matrix $\lambda_m(n)$ defined in (4.2.11). We define

$$J_+(n) = \begin{bmatrix} \mathrm{I}_{K'(n)} & 0 \\ 0 & 0 \end{bmatrix},\ J_-(n) = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{I}_{K'(n)} \end{bmatrix},\ J(n) = J_+(n) - J_-(n) \tag{4.3.18}$$

and set

$$\begin{gathered} \Pi^{\ell,+}_n = \begin{bmatrix} \Pi'^{\ell}_n & 0 \\ 0 & 0 \end{bmatrix},\ \Pi^{\ell,-}_n = \begin{bmatrix} 0 & 0 \\ 0 & \Pi'^{\ell}_n \end{bmatrix}, \\ \Pi^+_n = \begin{bmatrix} \Pi'_n & 0 \\ 0 & 0 \end{bmatrix},\ \Pi^-_n = \begin{bmatrix} 0 & 0 \\ 0 & \Pi'_n \end{bmatrix}, \end{gathered} \tag{4.3.19}$$

so that $\Pi_n = \Pi^+_n + \Pi^-_n$ and, denoting by D_0 the operator with symbol $D_0(n)$ given by (4.2.23), (4.2.24),

$$D_0 \Pi^{\ell,\pm}_n = \pm i\omega_m(n,\ell)\Pi^{\ell,\pm}_n. \tag{4.3.20}$$

By (4.3.8), we have also

$$D'_0 \Pi^{\pm}_n = \pm i\omega_m(n)\Pi^{\pm}_n. \tag{4.3.21}$$

Remind the map $\mathcal{F}_n : L^2(\mathbb{S}^1;\mathbb{K}^2) \to \mathbb{K}^{K(n)}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) defined by (2.1.6) and set

$$\widetilde{\Pi}^{\ell,\pm}_n = \mathcal{F}_n \circ \Pi^{\ell,\pm}_n \circ \mathcal{F}^*_n,\ \widetilde{\Pi}^{\pm}_n = \mathcal{F}_n \circ \Pi^{\pm}_n \circ \mathcal{F}^*_n. \tag{4.3.22}$$

These are projectors on $\mathbb{K}^{K(n)}$ and we have

$$
\begin{aligned}
D_0(n)\widetilde{\Pi}_n^{\ell,\pm} &= \widetilde{\Pi}_n^{\ell,\pm} D_0(n) = \pm i\omega_m(n,\ell)\widetilde{\Pi}_n^{\ell,\pm}\\
D_0'(n)\widetilde{\Pi}_n^{\pm} &= \widetilde{\Pi}_n^{\pm} D_0'(n) = \pm i\omega_m(n)\widetilde{\Pi}_n^{\pm}.
\end{aligned}
\tag{4.3.23}
$$

Let $c^1 \in \Sigma^{d,\nu}_{p,N_0}(N)$ be such that $L(c^1)$ vanishes identically. Compose $L(c^1)$ (given by the left hand side of (4.3.6)) on the left by $\widetilde{\Pi}_{n_0}^{\ell_0,\epsilon_0}$ and on the right by $\widetilde{\Pi}_{n_{p+1}}^{\ell_{p+1},\epsilon_{p+1}}$, and evaluate it at

$$
\Pi_{n'}^{\ell',\epsilon'}U' = (\Pi_{n_1}^{\ell_1,\epsilon_1}u_1, \dots, \Pi_{n_p}^{\ell_p,\epsilon_p}u_p)
$$

where $\epsilon_j \in \{+,-\}$ $j = 0, \dots, p+1$. We get

$$
\begin{aligned}
\sum_{j=1}^{p} \widetilde{\Pi}_{n_0}^{\ell_0,\epsilon_0} c^1(P_0\Pi_{n_1}^{\ell_1,\epsilon_1}u_1, \dots, P_0 D_0 \Pi_{n_j}^{\ell_j,\epsilon_j}u_j, \dots, P_0\Pi_{n_p}^{\ell_p,\epsilon_p}u_p; n_0, n_{p+1})\widetilde{\Pi}_{n_{p+1}}^{\ell_{p+1},\epsilon_{p+1}}&\\
+\widetilde{\Pi}_{n_0}^{\ell_0,\epsilon_0} c^1(P_0\Pi_{n_1}^{\ell_1,\epsilon_1}u_1, \dots, P_0\Pi_{n_p}^{\ell_p,\epsilon_p}u_p; n_0, n_{p+1}) D_0(n_{p+1})\widetilde{\Pi}_{n_{p+1}}^{\ell_{p+1},\epsilon_{p+1}}&\\
-\widetilde{\Pi}_{n_0}^{\ell_0,\epsilon_0} D_0(n_0) c^1(P_0\Pi_{n_1}^{\ell_1,\epsilon_1}u_1, \dots, P_0\Pi_{n_p}^{\ell_p,\epsilon_p}u_p; n_0, n_{p+1})\widetilde{\Pi}_{n_{p+1}}^{\ell_{p+1},\epsilon_{p+1}} &\equiv 0.
\end{aligned}
$$

Using (4.3.20), (4.3.23) we may write this as

$$
i\Big(\sum_{j=1}^{p+1} \epsilon_j\omega_m(n_j,\ell_j) - \epsilon_0\omega_m(n_0,\ell_0)\Big)\widetilde{\Pi}_{n_0}^{\ell_0,\epsilon_0}\tilde{c}^1(\Pi_{n'}^{\ell',\epsilon'}U'; n_0, n_{p+1})\widetilde{\Pi}_{n_{p+1}}^{\ell_{p+1},\epsilon_{p+1}} \equiv 0.
$$

Condition (4.3.2) shows that for m outside $\mathcal{N}$, the scalar coefficient above never vanishes, which implies $\tilde{c}^1 \equiv 0$, whence $c^1 \equiv 0$. This proves (i) of the lemma.

To prove (ii), we remark that if $a \in F$ is given, we may define $c^1 \in F$ with $L(c^1) = a$ by

$$
\begin{aligned}
&\widetilde{\Pi}_{n_0}^{\ell_0,\epsilon_0}\tilde{c}^1(\Pi_{n'}^{\ell',\epsilon'}U'; n_0, n_{p+1})\widetilde{\Pi}_{n_{p+1}}^{\ell_{p+1},\epsilon_{p+1}} =\\
&- i\Big(\sum_{j=1}^{p+1} \epsilon_j\omega_m(n_j,\ell_l) - \epsilon_0\omega_m(n_0,\ell_0)\Big)^{-1}\widetilde{\Pi}_{n_0}^{\ell_0,\epsilon_0}\tilde{a}(\Pi_{n'}^{\ell',\epsilon'}U'; n_0, n_{p+1})\widetilde{\Pi}_{n_{p+1}}^{\ell_{p+1},\epsilon_{p+1}}.
\end{aligned}
$$

Since by definition of F, n_0, n_{p+1} stay in a bounded set of indices, the estimates of definition of a symbol hold true trivially. $\square$

Proof of proposition 4.3.2: Using notations (4.3.13), (4.3.15), we shall construct operators

$$
\begin{aligned}
L^{-1} &: \Sigma'^{d,\nu}_{p,N_0} \to \Sigma'^{d,\nu+N_0}_{p,N_0}\\
L^{-1} &: \Sigma''^{d,\nu}_{p,N_0} \to \Sigma''^{d-1,\nu+N_0}_{p,N_0}
\end{aligned}
\tag{4.3.24}
$$

such that $L \circ L^{-1} = \mathrm{Id}$. This will give the wanted conclusion. It will be enough to construct for any N

$$
\begin{aligned}
L_N^{-1} &: \Sigma'^{d,\nu}_{p,N_0}(N) \to \Sigma'^{d,\nu+N_0}_{p,N_0}(N+1)\\
L_N^{-1} &: \Sigma''^{d,\nu}_{p,N_0}(N) \to \Sigma''^{d-1,\nu+N_0}_{p,N_0}(N)
\end{aligned}
\tag{4.3.25}
$$

such that $L \circ L_N^{-1} : \Sigma'^{d,\nu}_{p,N_0}(N) \to \Sigma'^{d,\nu+N_0+1}_{p,N_0}(N)$ and $L \circ L_N^{-1} : \Sigma''^{d,\nu}_{p,N_0}(N) \to \Sigma''^{d,\nu+N_0}_{p,N_0}(N)$ coincide with identity. Actually, since L is injective by lemma 4.3.3,

$$L_N^{-1}|_{\Sigma'^{d,\nu}_{p,N_0}(N+1)} = L_{N+1}^{-1},\ L_N^{-1}|_{\Sigma''^{d,\nu}_{p,N_0}(N+1)} = L_{N+1}^{-1}$$

which allows us to define L^{-1} satisfying (4.3.24).

If $A_N > 0$ is a constant to be chosen, we decompose

$$\Sigma'^{d,\nu}_{p,N_0}(N) = F'_N \oplus \Sigma'^{d,\nu}_{p,N_0}(N, A_N),\ \Sigma''^{d,\nu}_{p,N_0}(N) = F''_N \oplus \Sigma''^{d,\nu}_{p,N_0}(N, A_N)$$

where F'_N, F''_N is the subspace made of symbols a satisfying $a(\cdot; n_0, n_{p+1}) \equiv 0$ for $n_0+n_{p+1} > A_N$. By (ii) of lemma 4.3.3, it is enough to construct

$$\begin{aligned} L_N^{-1} &: \Sigma'^{d,\nu}_{p,N_0}(N, A_N) \to \Sigma'^{d,\nu+N_0}_{p,N_0}(N+1, A_N) \\ L_N^{-1} &: \Sigma''^{d,\nu}_{p,N_0}(N, A_N) \to \Sigma''^{d-1,\nu+N_0}_{p,N_0}(N, A_N) \end{aligned} \tag{4.3.26}$$

for A_N large enough. Remind decomposition (4.3.9) of L, and let us construct first an inverse $L_{0,N}^{-1}$ to L_0. We take $\tilde{a}$ respectively in $\Sigma'^{d,\nu}_{p,N_0}(N, A_N)$ or $\Sigma''^{d,\nu}_{p,N_0}(N, A_N)$ and look for c^1 in the right hand side of (4.3.26) with $L_0(c^1) = \tilde{a}$. We use expressions (4.3.12), (4.3.14) for $L_0(c^1)$. If we compose on the right with $J_{\epsilon_{p+1}}$ defined in (4.3.18) and evaluate $L_0(c^1)$ at $\Pi_{n'}^{\ell',\epsilon'} U' = (\Pi_{n_1}^{\ell_1,\epsilon_1} u_1, \dots, \Pi_{n_p}^{\ell_p,\epsilon_p} u_p)$, we get respectively the equalities

$$\begin{aligned} \tilde{c}^1_{D_0}(\Pi_{n'}^{\ell',\epsilon'} U'; n_0, n_{p+1}) J_{\epsilon_{p+1}} + \tilde{c}^1(\Pi_{n'}^{\ell',\epsilon'} U'; n_0, n_{p+1}) (D'_0(n_{p+1}) \mp D'_0(n_0)) J_{\epsilon_{p+1}} \\ = \tilde{a}(\Pi_{n'}^{\ell',\epsilon'} U'; n_0, n_{p+1}) J_{\epsilon_{p+1}}. \end{aligned}$$

Using (4.3.7) and (4.3.20), (4.3.23) we see that we may define c^1 by

$$\tilde{c}^1(\Pi_{n'} U'; n_0, n_{p+1}) = - \sum_{(\ell_1,\epsilon_1),\dots,(\ell_p,\epsilon_p),\epsilon_{p+1}} i F_{\mp}^{\ell',\epsilon}(n_0, \dots, n_{p+1})^{-1} \tilde{a}(\Pi_{n'}^{\ell',\epsilon'} U'; n_0, n_{p+1}) J_{\epsilon_{p+1}} \tag{4.3.27}$$

where the sum is taken for $\ell_1, \dots, \ell_p, \epsilon_1, \dots, \epsilon_{p+1}$ in a set of bounded cardinal, and where

$$F_{\mp}^{\ell',\epsilon}(n_0, \dots, n_{p+1}) = \sum_{j=1}^{p} \epsilon_j \omega_m(n_j, \ell_j) + \epsilon_{p+1}(\omega_m(n_{p+1}) \mp \omega_m(n_0)).$$

It is enough to check that each term in the sum (4.3.27) belongs to the right hand side of (4.3.26). Remark that $F_{-}^{\ell',\epsilon}$ is a function of type (3.3.10) that satisfies (3.3.11): if $|n_0 - n_{p+1}|$ is large relatively to $|n'|$, this follows from the fact that $\omega_m(n) = n + O(1/n), n \to +\infty$. If $|n_0 - n_{p+1}| \leq C|n'|$ this is a consequence of proposition 4.3.1. By inequalities (3.3.12), in the case of sign $-$ in (4.3.27), we see that $\tilde{c}^1$ is a symbol in $\Sigma'^{d,\nu+N_0}_{p,N_0}(N+1, A_N)$ (taking eventually for N_0 a larger value than the one of (3.3.11)). In the case of $F_{+}^{\ell',\epsilon}$, we remark that it satisfies (3.3.13). So (3.3.12) will be controlled in terms of (3.3.14). This implies that for $\tilde{a} \in \Sigma''^{d,\nu}_{p,N_0}(N, A_N)$, (4.3.27) defines a symbol $\tilde{c}^1$ in $\Sigma''^{d-1,\nu+N_0}_{p,N_0}(N, A_N)$. Consequently we have defined a bounded inverse $L_{0,N}^{-1}$ to L_0, acting on space (4.3.26). To define L_N^{-1} as

$$L_N^{-1} = (L_0(\mathrm{Id} + L_{0,N}^{-1} L_1))^{-1} = (\mathrm{Id} + L_{0,N}^{-1} L_1)^{-1} L_{0,N}^{-1}$$

we just need to check that the operator norm of $L_{0,N}^{-1} \circ L_1$ from $\Sigma'^{d,\nu+N_0}_{p,N_0}(N+1, A_N)$ (resp. $\Sigma''^{d-1,\nu+N_0}_{p,N_0}(N, A_N)$) to itself is smaller than one if A_N is large enough. But we have seen that L_1 sends $\Sigma^{d',\nu}_{p,N_0}(N)$ to $\Sigma^{-\infty,0}_{p,N_0}(N)$ for any d'. By definition (4.3.11) of L_1, the same is true for the Σ' or Σ'' spaces, so the operator norm of $L_{0,N}^{-1} \circ L_1$ on the above spaces is bounded from above by C_N/A_N, where $C_N > 0$ in independent of A_N (it suffices to extract from the gain on the order coming from L_1 a factor $\frac{1}{n_0+n_{p+1}} \leq \frac{1}{A_N}$). The conclusion follows for large enough A_N. $\square$

We shall need also a result, similar to proposition 4.3.2, but for remainder operators.

Proposition 4.3.4 *Let $d \in \mathbb{R}$, let $p \in \mathbb{N}$ an odd number and $\nu \in \mathbb{R}_+$. For every $\widetilde{M} \in \mathcal{R}^{d,\nu}_{p+1}$ there is $\widetilde{M}_1 \in \mathcal{R}^{d,\nu+N_0}_{p+1}$ such that for any $u_1, \dots, u_{p+1} \in \mathcal{E}$*

$$\sum_{j=1}^{p+1} \widetilde{M}_1(u_1, \dots, M_0 u_j, \dots, u_{p+1}) + M_0^* \widetilde{M}_1(u_1, \dots, u_{p+1}) = \widetilde{M}(u_1, \dots, u_{p+1}). \tag{4.3.28}$$

Proof: We extend $\widetilde{M}, \widetilde{M}_1$ as $\mathbb{C}$-multilinear maps, replace u_j bu $P_0 u_j$ and compose on the left by P_0^*. Since $M_0 P_0 = P_0 D_0$ and $D_0^* = -D_0$ we get

$$\begin{aligned} \sum_{j=1}^{p+1} P_0^* \widetilde{M}_1(P_0 u_1, \dots, P_0 D_0 u_j, \dots, P_0 u_{p+1}) - D_0 P_0^* \widetilde{M}_1(P_0 u_1, \dots, P_0 u_{p+1}) = \\ P_0^* \widetilde{M}(P_0 u_1, \dots, P_0 u_{p+1}). \end{aligned} \tag{4.3.29}$$

We use notations (4.3.19). We compose on the left (4.3.29) with $\Pi^{\ell_0,\epsilon_0}_{n_0}$ and replace u_j by $\Pi^{\ell_j,\epsilon_j}_{n_j} u_j$, for any possible values of $n_0, \dots, n_{p+1}$, $\ell_0, \dots, \ell_{p+1}$, $\epsilon_0, \dots, \epsilon_{p+1}$. If $U = (u_1, \dots, u_{p+1})$, $n = (n_1, \dots, n_{p+1})$, $\ell = (\ell_1, \dots, \ell_{p+1})$, $\epsilon = (\epsilon_1, \dots, \epsilon_{p+1})$ we set

$$\Pi^{\ell,\epsilon}_n U = (\Pi^{\ell_1,\epsilon_1}_{n_1} u_1, \dots, \Pi^{\ell_{p+1},\epsilon_{p+1}}_{n_{p+1}} u_{p+1}).$$

Using (4.3.20) we see that (4.3.29) may be written

$$\Pi^{\ell_0,\epsilon_0}_{n_0} P_0^* \widetilde{M}_1(P_0 \Pi^{\ell,\epsilon}_n U) = -i \Big(\sum_{j=1}^{p+1} \epsilon_j \omega_m(n_j, \ell_j) - \epsilon_0 \omega_m(n_0, \ell_0) \Big)^{-1} \Pi^{\ell_0,\epsilon_0}_{n_0} P_0^* \widetilde{M}(\Pi^{\ell,\epsilon}_n P_0 U)$$

so that replacing U by $P_0^{-1} U$

$$\Pi_{n_0} \widetilde{M}_1(\Pi_n U) = -i \sum_{(\ell_0,\epsilon_0),\dots,(\ell_{p+1},\epsilon_{p+1})} \Big(\sum_{j=1}^{p+1} \epsilon_j \omega_m(n_j, \ell_j) - \epsilon_0 \omega_m(n_0, \ell_0) \Big)^{-1} (P_0^*)^{-1} \Pi^{\ell_0,\epsilon_0}_{n_0} P_0^* \widetilde{M}(\Pi^{\ell,\epsilon}_n U)$$

where the sum is taken for $\ell_0, \dots, \ell_{p+1}$, $\epsilon_0, \dots, \epsilon_{p+1}$ staying in a bounded set of indices. By proposition 4.3.1 the first factor in the sum is bounded from above by $C\mu(n_0, \dots, n_{p+1})^{N_0}$. If we use that $\widetilde{M}$ satisfies estimates of type (2.1.15) the same is true for $\widetilde{M}_1$, with ν replaced by $\nu + N_0$, since $\mu(n_0, \dots, n_{p+1}) \leq \max_2(n_1, \dots, n_{p+1})$. This concludes the proof. $\square$

Proof of theorem 1.1.1: We wrote equation (1.1.4) under the equivalent form (4.2.4) or (4.2.13). It is enough to show that there is s_0 large enough, such that if $s \geq s_0$, there is $C_s > 0$ and $R_0 > 0$ so that, if $u(t, \cdot)$ is a solution of (4.2.4) defined on some interval $[0, T]$, with Cauchy data in H^s, one has for any $t \in [0, T]$

$$\|u(t,\cdot)\|^2_{H^s} \leq C_s \big[\|u(0,\cdot)\|^2_{H^s} + \int_0^t \|u(\tau,\cdot)\|^{r-1}_{H^{s_0}} \|u(\tau,\cdot)\|^2_{H^s}\, d\tau\big] \tag{4.3.30}$$

as long as $\|u(t,\cdot)\|_{H^{s_0}} \leq R_0$. Actually, applying (4.3.30) with $s = s_0$, assuming $\|u(0,\cdot)\|_{H^{s_0}} \leq \epsilon$ and taking $R_0 = (2C_{s_0})^{1/2}\epsilon$, we see that we may extend the solution as an H^{s_0} function up to time $t_0 = \frac{1}{2C_{s_0}} R_0^{-r+1} = c_{s_0}\epsilon^{-r+1}$. If the Cauchy data are H^s with $s \geq s_0$, the solution is also in H^s on the same interval. It will be bounded in H^s on an interval of length $\frac{1}{2C_s} R_0^{-r+1} = c_s \epsilon^{-r+1}$.

Because of (4.2.16), we may in (4.3.30) replace $\|u(t,\cdot)\|^2_{H^s}$ by $\Theta^s_0(u(t,\cdot))$. Moreover, if in the right hand side of (4.2.17) we replace $\mathrm{Op}(c(u;\cdot))$ by

$$\frac{1}{2}[\mathrm{Op}(c(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot)) + \mathrm{Op}((1+a_\chi)(u;\cdot))\mathrm{Op}(c(u;\cdot))] \tag{4.3.31}$$

we make appear an error that may be written by corollary 3.3.5 (i) $\langle \mathrm{Op}(e(u;\cdot))\tilde{u}, \tilde{u}\rangle$, where e is a symbol in $\widetilde{\Sigma}^{2s,\nu'}_1$ for some ν' independent of s, of valuation $v(e) \geq 2\kappa \geq r-1$. Consequently, by proposition 2.1.3 and (4.2.22)

$$\|\mathrm{Op}(e(u;\cdot))\tilde{u}\|_{H^{-s}} \leq C\|u\|^{r-1}_{H^{s_0}}\|u\|_{H^s}$$

if $s \geq s_0$, and s_0 is large enough relatively to ν'. We thus see that if we modify the definition of Θ^s_0 replacing in (4.2.17) $\mathrm{Op}(c(u;\cdot))$ by (4.3.31), we still get a quantity equivalent to $\|u\|^2_{H^s}$ when $\|u\|_{H^{s_0}}$ is small enough. We may thus assume from now on that

$$\Theta^s_0(u) = \frac{1}{2}\langle[\mathrm{Op}(c^0(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot)) + \mathrm{Op}((1+a_\chi)(u;\cdot))^*\mathrm{Op}(c^0(u;\cdot))]\tilde{u}, \tilde{u}\rangle \tag{4.3.32}$$

for a scalar self adjoint symbol $c^0 \in \widetilde{\Sigma}^{2s,\nu}_1$, and c^0 satisfying condition $C(\kappa, r)$ of definition 3.3.2. We may decompose c^0 as a finite sum of homogeneous symbols $c^0_p \in \Sigma^{2s,\nu}_{p,1}$. Remark that the contributions coming from the components homogeneous of degree $p \geq r-1$ give again a contribution to $\Theta^s_0(u)$ which is $O(\|u\|^{r-1}_{H^{s_0}}\|u\|^2_{H^s})$. Modifying again the definition of Θ^s_0, we may thus assume

$$c^0 = \sum_{p=0}^{r-2} c^0_p.$$

Since c^0 satisfies $C(\kappa, r)$, terms indexed by even p's in the above sum are zero. We compute the time derivative of (4.3.32) applying to each homogeneous component proposition 4.2.4. Remark that assumptions (4.2.34), (4.2.35) are satisfied since c^0 is scalar and self-adjoint. We get, by (4.2.39)

$$\begin{aligned}\frac{d}{dt}\Theta^s_0(u(t,\cdot)) =& \langle \mathrm{Op}(c^0_{M_0}(u;\cdot))\tilde{u}, \tilde{u}\rangle + \langle[\mathrm{Op}(c^0(u;\cdot))D_0 + D_0^*\mathrm{Op}(c^0(u;\cdot))]\tilde{u}, \tilde{u}\rangle\\ &+ \langle \mathrm{Op}(e^0(u;\cdot))\tilde{u}, \tilde{u}\rangle + 2\mathrm{Re}\,\langle R^0(u), u\rangle\end{aligned}$$

where $e^0 \in \widetilde{\Sigma}^{2s,\nu'}_{N_0}$, $R^0 \in \widetilde{\mathcal{R}}^{2s,\nu'}$ for some ν' independent of s, and with $v(e^0) \geq \kappa$, $v(R^0) \geq \kappa + 1$. Moreover e^0 is self-adjoint and these symbols and operators satisfy condition $C(\kappa, r)$. Since c^0 is scalar and $D_0^* = -D_0$, we get from corollary 3.3.5 (ii) that the second duality bracket may be written $\langle \mathrm{Op}(b(u;\cdot))\tilde{u}, \tilde{u}\rangle$ for a symbol $b \in \widetilde{\Sigma}^{2s,\nu}_1$ for some ν independent of s. Moreover, since c^0 satisfies condition $C(\kappa, r)$, $c^0_{M_0}$ and b have valuation larger or equal to κ, and verify also $C(\kappa, r)$. We may thus write

$$\frac{d}{dt}\Theta^s_0(u(t,\cdot)) = \langle \mathrm{Op}(g(u;\cdot))\tilde{u}, \tilde{u}\rangle + 2\mathrm{Re}\,\langle R^0(u), u\rangle \tag{4.3.33}$$

for a new symbol $g \in \widetilde{\Sigma}^{2s,\nu}_1$ with $v(g) \geq \kappa$, g satisfying condition $C(\kappa, r)$. In particular, the homogeneous components of order p of g with $\kappa \leq p < r-1$ vanish if p is even. Moreover we may assume g self-adjoint. For odd $p, \kappa \leq p < r-1$, we decompose the corresponding contribution g_p as $g'_p + g''_p$, where g'_p satisfies assumption (H_D) and g''_p satisfies (H_{ND}). By proposition 4.3.2, for each such p, we may find $c^{1\prime}_p \in \Sigma^{2s,\nu'}_{p,N_0}$, $c^{1\prime\prime}_p \in \Sigma^{2s-1,\nu'}_{p,N_0}$ for some ν' independent of s, such that (4.3.5) holds true for $c^1_p = c^{1\prime}_p + c^{1\prime\prime}_p$, when its right hand side is replaced by g_p. In particular, these c^1_p have the structure (4.2.34), (4.2.35) which allows us to apply proposition 4.2.4. More precisely, define

$$\Theta^s_1(u) = \frac{1}{2}\sum_{\substack{\kappa\leq p<r-1\\ p \text{ odd}}} \langle [\mathrm{Op}(c^1_p(u;\cdot))\mathrm{Op}((1+a_\chi)(u;\cdot)) + \mathrm{Op}((1+a_\chi)(u;\cdot))^*\mathrm{Op}(c^1_p(u;\cdot))]\tilde{u}, \tilde{u}\rangle.$$

By (4.2.39) and (4.3.5) we have

$$\frac{d}{dt}\Theta^s_1(u(t,\cdot)) = \langle \mathrm{Op}(g(u;\cdot))\tilde{u}, \tilde{u}\rangle + \langle \mathrm{Op}(f^0(u;\cdot))u, u\rangle + 2\mathrm{Re}\,\langle S^0(u), u\rangle \tag{4.3.34}$$

where $f^0 \in \widetilde{\Sigma}^{2s,\nu'}_{N_0}$, $S^0 \in \widetilde{\mathcal{R}}^{2s,\nu'}$ for some ν' independent of s, with $v(f^0) \geq 2\kappa$, $v(S^0) \geq 2\kappa + 1$. (We used again (4.2.20) to express $\tilde{u}$ in terms of u in the last but one term coming from (4.2.39)).

Let us define also a perturbation to get rid of the $\langle R^0(u), u\rangle$ term in (4.3.33). We may decompose $R^0 = R'^0 + \tilde{R}^0$ with $R'^0 = \sum_{\kappa\leq p<r-1} R^0_{p+1}$ and $\tilde{R}^0 \in \widetilde{\mathcal{R}}^{2s,\nu}$ of valuation larger or equal to r, and where $R^0_{p+1} \in \mathcal{R}^{2s,\nu'}_{p+1}$ and the sum is indexed by odd p (since R^0 satisfies condition $C(\kappa, r)$). Define $\widetilde{M}_{p+1}$ as the solution of equation (4.3.28), when the right hand side is replaced by R^0_{p+1}. Then $\widetilde{M}_{p+1} \in \mathcal{R}^{2s,\nu'+N_0}_{p+1}$ and if we set

$$\Theta^s_2(u) = 2\mathrm{Re}\sum_{\substack{\kappa\leq p<r-1\\ p \text{ odd}}} \langle \widetilde{M}_{p+1}(u,\dots,u), u\rangle$$

it follows from proposition 4.2.5 that

$$\frac{d}{dt}\Theta^s_2(u(t,\cdot)) = 2\mathrm{Re}\left[\langle R'^0(u), u\rangle + \langle R^1(u), u\rangle + \sum_{\substack{\kappa\leq p<r-1\\ p \text{ odd}}} \langle \widetilde{M}_{p+1}(u,\dots,u), R^2_{p+1}(u)\rangle\right] \tag{4.3.35}$$

where $R^1 \in \widetilde{\mathcal{R}}^{2s+1,\nu''}$, $R^2_{p+1} \in \widetilde{\mathcal{R}}^{0,\nu''}$ for some ν'' independent of s and $v(R^1) \geq 2\kappa + 1 \geq r$, $v(R^2_{p+1}) \geq \kappa + 1$. Combining (4.3.33), (4.3.34), (4.3.35) we get

$$\begin{aligned}\frac{d}{dt}[\Theta_0^s(u(t,\cdot)) &- \Theta_1^s(u(t,\cdot)) - \Theta_2^s(u(t,\cdot))] = \\ &- \langle \mathrm{Op}(f^0(u;\cdot))u, u\rangle + 2\mathrm{Re}\, \langle \tilde{R}^0(\tilde{u}) - S^0(u) - R^1(u), u\rangle \\ &- 2\mathrm{Re} \sum_{\substack{\kappa \leq p < r-1 \\ p \text{ odd}}} \langle \widetilde{M}_{p+1}(u, \dots, u), R^2_{p+1}(u)\rangle.\end{aligned} \tag{4.3.36}$$

The right hand side is bounded from above by

$$\begin{aligned}C[\|\mathrm{Op}(f^0(u;\cdot))u\|_{H^{-s}}\|u\|_{H^s} &+ \|\tilde{R}^0(\tilde{u}) - S^0(u) - R^1(u)\|_{H^{-s}}\|u\|_{H^s} \\ &+ \sum_{\substack{\kappa \leq p < r-1 \\ p \text{ odd}}} \|\widetilde{M}_{p+1}(u, \dots, u)\|_{H^{-s}}\|R^2_{p+1}(u)\|_{H^s}].\end{aligned} \tag{4.3.37}$$

By proposition 2.1.3, and using that $v(f^0) \geq 2\kappa$, there is some s_0, depending on ν' but not on s, such that when $s \geq s_0$ the first term in (4.3.37) is bounded by $C\|u\|^{2\kappa}_{H^{s_0}}\|u\|^2_{H^s}$, as long as $\|u\|_{H^{s_0}} \leq 1$. In the second term of (4.3.37), $S^0, R^1, \tilde{R}^0$ belong to $\widetilde{\mathcal{R}}^{2s+1,\nu''}$ for some ν'' independent of s, and have valuation larger or equal to r. By lemma 2.1.7 and inequality (2.1.19), for s large enough relatively to ν'', the second term in (4.3.37) is controlled by $C\|u\|^{r-1}_{H^{s_0}}\|u\|^2_{H^s}$. Since $R^2_{p+1} \in \widetilde{\mathcal{R}}^{0,\nu''}$ with $v(R^2_{p+1}) \geq \kappa + 1$, lemma 2.1.7 implies $\|R^2_{p+1}(u)\|_{H^s} \leq C\|u\|^{\kappa}_{H^{s_0}}\|u\|_{H^s}$ for some s_0 large enough. Since $\widetilde{M}_{p+1} \in \mathcal{R}^{2s,\nu''}_{p+1}$ with ν'' independent of s and $p+1 \geq \kappa+1$, the same lemma gives the estimate $\|\widetilde{M}_{p+1}(u, \dots, u)\|_{H^{-s}} \leq C\|u\|^{\kappa}_{H^{s_0}}\|u\|_{H^s}$if s_0 is large enough (independently of s), and $\|u\|_{H^{s_0}} \leq 1$. Finally we get for (4.2.34) an upper bound in terms of

$$C\|u\|^{r-1}_{H^{s_0}}\|u\|^2_{H^s}.$$

using that $2\kappa \geq r - 1$. It then follows from (4.3.36) that for $t \geq 0$

$$\begin{aligned}&\Theta_0^s(u(t,\cdot)) - \Theta_1^s(u(t,\cdot)) - \Theta_2^s(u(t,\cdot)) \leq \\ &\Theta_0^s(u(0,\cdot)) - \Theta_1^s(u(0,\cdot)) - \Theta_2^s(u(0,\cdot)) + C\int_0^t \|u(\tau,\cdot)\|^{r-1}_{H^{s_0}}\|u(\tau,\cdot)\|^2_{H^s}\, d\tau\end{aligned} \tag{4.3.38}$$

when $s \geq s_0$ large enough and when for $0 \leq t' \leq t$, $\|u(t',\cdot)\|_{H^{s_0}} \leq 1$. Again by proposition 2.1.3 and lemma 2.1.7, we get when $\|u(t,\cdot)\|_{H^{s_0}} \leq 1$

$$|\Theta_1^s(u(t,\cdot))| + |\Theta_2^s(u(t,\cdot))| \leq C\|u(t,\cdot)\|^{\kappa}_{H^{s_0}}\|u(t,\cdot)\|^2_{H^s}. \tag{4.3.39}$$

Inequality (4.3.30) follows from (4.3.38), (4.3.39) when $\|u(t',\cdot)\|_{H^{s_0}}$ stays small enough on the interval $[0,t]$. This concludes the proof. □

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